Introduction to the h-Principle

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To

Vladimir Igorevich Arnold

who introduced us to the world of singularities

and

Misha Gromov

who taught us how to get rid of them

Contents

Figure 0.1. The relations between chapters of the book

Preface

A *partial differential relation* $\mathcal{R}$ is any condition imposed on the partial derivatives of an unknown function. A solution of $\mathcal{R}$ is any function which satisfies this relation.

The classical partial differential relations, mostly rooted in Physics, are usually described by (systems of) equations. Moreover, the corresponding systems of equations are mostly *determined*: the number of unknown functions is equal to the number of equations. Given appropriate boundary conditions, such a differential relation usually has a unique solution. In some cases this solution can be found using certain *analytical* methods (potential theory, Fourier method and so on).

In differential geometry and topology one often deals with systems of partial differential equations, as well as partial differential *inequalities*, which have infinitely many solutions whatever boundary conditions are imposed. Moreover, sometimes solutions of these differential relations are C^0-dense in the corresponding space of functions or mappings. The systems of differential equations in question are usually (but not necessarily) *underdetermined*. We discuss in this book *homotopical* methods for solving this kind of differential relations. Any differential relation has an underlying algebraic relation which one gets by substituting derivatives by new independent variables. A solution of the corresponding algebraic relation is called a *formal* solution of the original differential relation $\mathcal{R}$. Its existence is a necessary condition for the solvability of $\mathcal{R}$, and it is a natural starting point for exploring $\mathcal{R}$. Then one can try to deform the formal solution into a genuine solution. We say that the *h-principle* holds for a differential relation $\mathcal{R}$, if any formal solution of $\mathcal{R}$ can be deformed into a genuine solution.

The *notion* of h-principle (under the name "w.h.e.-principle") first appeared in [**Gr71**] and [**GE71**]. The *term* "h-principle" was introduced and popularized by M. Gromov in his book [**Gr86**]. The h-principle for solutions of partial differential relations exposed the soft/hard (or flexible/rigid) dichotomy for the problems formulated in terms of derivatives: a particular analytical problem is "soft" or "abides by the h-principle" if its solvability is determined by some underlying *algebraic* or *geometric* data. The softness phenomena was first discovered in the fifties by J. Nash [**Na54**] for isometric C^1-immersions, and by S. Smale [**Sm58, Sm59**] for differential immersions. However, instances of soft problems appeared earlier (e.g. H. Whitney's paper [**Wh37**]). In the sixties several new geometrically interesting examples of soft problems were discovered by M. Hirsch, V. Poénaru, A. Phillips, S. Feit and other authors (see [**Hi59**], [**Po66**], [**Ph67**], [**Fe69**]). In his dissertation [**Gr69**], the paper [**Gr73**] and later in his book [**Gr86**] Gromov transformed Smale's and Nash's ideas into two powerful general methods for solving partial differential relations: *continuous sheaves* (or the *covering homotopy*) method and the *convex integration* method. The third method, called *removal of singularities*, was first introduced and explored in [**GE71**].

There is an opinion that *"the h-principle is the hardest part of Gromov's work to popularize"* (see [**Be00**]). We wrote our book in order to improve the situation. We consider here two geometrical methods: *holonomic approximation*, which is a version of the method of *continuous sheaves*, and *convex integration.* We do not pretend to cover here the content of Gromov's book [**Gr86**], but rather want to prepare and motivate the reader to look for hidden treasures there. On the other hand, the reader interested in applications will find that with few notable exceptions (e.g. Lohkamp's theory [**Lo95**] of negative Ricci curvature and Donaldson's theory [**?**] of approximately holomorphic sections) most instances of the h-principle which are known today can be treated by the methods considered in the present book.

The first three parts of the book are devoted to a quite general theorem about holonomic approximation of sections of jet-bundles and its applications. Given an arbitrary submanifold $V_0 \subset V$ of positive codimension, the Holonomic Approximation Theorem allows us to solve any *open* differential relations $\mathcal{R}$ near a slightly perturbed submanifold $\widetilde{V}_0 = h(V)$ where $h : V \to V$ is a C^0-small diffeomorphism. Gromov's h-principle for open Diff V-invariant differential relations on open manifolds, his directed embedding theorem, as well as some other results in the spirit of the h-principle are immediate corollaries of the Holonomic Approximation Theorem.

The method for proving the h-principle based on the Holonomic Approximation Theorem works well for *open* manifolds. Applications to closed manifold require an additional trick, called *microextension*. It was first used by M. Hirsch in [**Hi59**]. The holonomic approximation method also works well for differential relations which are not open, but *microflexible*. The most interesting applications of this type come from Symplectic geometry. These applications are discussed in the third part of the book. For convenience of the reader the basic notions of Symplectic geometry are also reviewed in that part of the book.

The fourth part of the book is devoted to *convex integration theory*. Gromov's convex integration theory was treated in great detail by D. Spring in [**Sp98**]. In our exposition of convex integration we pursue a different goal. Rather than considering the sophisticated advanced version of convex integration presented in [**Gr86**], we explore only its simple version for first order differential relations, similar to the first exposition of the theory by Gromov in [**Gr73**]. Nevertheless, we prove here practically all the most interesting corollaries of the theory, including the Nash-Kuiper theorem on C^1-isometric embeddings.

Let us list here some available books and survey papers about the h-principle. Besides Gromov's book [**Gr86**], these are: Spring's book [**Sp98**], Adachi's book [**Ad84**], Haefliger's paper [**Ha71**], Poénaru's paper [**Po71**] and most recently Geiges' notes [**Ge01**].

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Intrigue

◀ **Examples**

A. Immersions. A smooth map $f : V \to W$ of an n-dimensional manifold V into a q-dimensional W, $n \leq q$, is called an *immersion* if its differential has the maximal rank n at every point. Two immersions are called *regularly homotopic* if one can be deformed to the other through a smooth family of immersions.

A1. For an immersion $f : S^1 \to \mathbb{R}^2$ we denote by $G(f)$ its *tangential degree*, i.e. the degree of the corresponding Gaussian map $S^1 \to S^1$. Then *two immersions $f, g : S^1 \to \mathbb{R}^2$ are regularly homotopic if and only if $G(f) = G(g)$*, see [**Wh37**] and Section 6.1 below.

A2. On the other hand, *any two immersions $S^2 \to \mathbb{R}^3$ are regularly homotopic*, see [**Sm58**] and Section 4.2 below. In particular, the standard 2-sphere in $\mathbb{R}^3$ can be inverted inside out through a family of immersions.

A3. Consider now pairs of immersions $(f_0, f_1) : D^2 \to \mathbb{R}^2$ which coincides near the boundary circle ∂D^2. What is the classification of such pairs up to the regular homotopy in this class? The answer turns out to be quite unexpected: *there are precisely two regular homotopy classes of such pairs. One is represented by the pair (j, j) where j is the inclusion $D^2 \hookrightarrow \mathbb{R}^2$, the second one is represented by the pair (f, g), where the immersions f and g are shown on Fig.0.2..* See [**El72**].

B. Isometric C^1-immersions. Is there a regular homotopy $f_t : S^2 \to \mathbb{R}^3$ which begins with the inclusion f_0 of the unit sphere and ends with an isometric immersion f_1 into the ball of radius $\frac{1}{2}$? Here the word isometric means *preserving length of all curves.* The answer is, of course, negative if f_1 is required to be C^2-smooth. Indeed, in this case the Gaussian curvature of the metric on S^2 should be ≥ 4 at least somewhere. However, surprisingly

Figure 0.2. The immersions f and g.

the answer is "yes" in the case of C^1*-immersions* (when the curvature is not defined but the curve length is), see [**Na54, Ku55**] and Chapter 21 below.

C. Mappings with a prescribed Jacobian. *Let* Ω *be a* n*-form on a closed oriented stably parallelizable* n*-dimensional manifold* M *such that* $\int_M \Omega = 0$*, and*

$$\eta = dx_1 \wedge \cdots \wedge dx_n$$

be the standard volume form on $\mathbb{R}^n$*. Then there exists a map* $f : M \to \mathbb{R}^n$ *such that* $f^*\eta = \Omega$*.* See [**GE73**]. ▶

All the above statements are examples of the *homotopy principle*, or the *h-principle*. Despite the fact that all these problems are asking for the solution of certain differential equations or inequalities, they can be reduced to problems of a pure homotopy-theoretic nature which then can be dealt with the methods of Algebraic topology. For instance, the regular homotopy classification of immersions $S^2 \to \mathbb{R}^3$ can be reduced to the computation of the homotopy group $\pi_2(\mathbb{R}P^3)$, which is trivial.

We are teaching in this book how to deal with these problems. In particular, two general methods which we describe here will be sufficient to handle all the above examples, except **A3** and **C**. In our sequel book "The h-principle and singularities" we will discuss other methods which prove, in particular, the two remaining results.

Another, maybe even more important, goal of this book is to teach the reader *how to recognize* the problems which may satisfy the h-principle. Of course, in the most interesting cases this is a very difficult question. As we will see below there are plenty of open problems where one neither can establish the h-principle, nor find a single instance of *rigidity*. Nevertheless we are confident that the reader should develop a pretty good intuition for the problems which may satisfy the h-principle.

Here are few more examples when the h-principle holds, fails or unknown.

◀ **Examples**

D. Totally real, Lagrangian and ε-Lagrangian embeddings. Let $T^2 = (\mathbb{R}/\mathbb{Z}) \times (\mathbb{R}/\mathbb{Z})$ be the 2-torus with the cyclic coordinates $x_1, x_2 \in \mathbb{R}/\mathbb{Z}$. Given an embedding $f : T^2 \to \mathbb{C}^2$ consider the vectors

$$v_1(x) = \frac{\partial f}{\partial x_1}(x) \text{ and } v_2(x) = \frac{\partial f}{\partial x_2}(x)\,, \quad x \in T^2\,.$$

The embedding f is called *real* or *totally real* if these vectors are linearly independent (over $\mathbb{C}$) for each $x \in T^2$. It is called *Lagrangian* if the *real* planes generated by the vectors $v_1(x), v_2(x)$ and $iv_1(x), iv_2(x)$ are orthogonal for each $x \in T^2$. For $0 < \varepsilon \leq \frac{\pi}{2}$ an embedding f is called ε-*Lagrangian* if the angle between these planes is greater than $\frac{\pi}{2} - \varepsilon$ for each $x \in T^2$. Thus Lagrangian are real, and real coincide with $(\pi/2)$-Lagrangian embeddings. Let us denote by N_{42} the space of real 4×2 matrices of rank 2. Identifying $\mathbb{C}^2$ with $\mathbb{R}^4$ we can view a 2×2 complex matrix as a 4×2 real matrix, and thus consider $\mathrm{GL}(2, \mathbb{C})$ as a subspace of N_{42}. With any embedding $f : T^2 \to \mathbb{C}^2$ we associate the map $v_f : T^2 \to N_{42}$ defined by the formula $v_f(x) = (v_1(x), v_2(x)) \in N_{42}$. If f is real then the image $v_f(T^2)$ is contained in $\mathrm{GL}(2, \mathbb{C})$.

D1. Real and, moreover, ε-Lagrangian embeddings satisfy the h-principle. Namely, *let $f : T^2 \to \mathbb{C}^2$ be any embedding. Suppose that the map $v_f : T^2 \to N_{4,2}$ is homotopic to a map $w : T^2 \to \mathrm{GL}(2, \mathbb{C}) \subset N_{4,2}$. Then for any $\varepsilon > 0$ the embedding f is isotopic to an ε-Lagrangian embedding. Any two ε-Lagrangian embeddings $f, g : T^2 \to \mathbb{C}^2$ such that the maps f and g are isotopic and the maps*

$$v_f, v_g : T^2 \to \mathrm{GL}(2, \mathbb{C})$$

are homotopic inside $\mathrm{GL}(2, \mathbb{C})$ *are isotopic via an ε-Lagrangian isotopy.* See [**Gr86**] and Section 19.3 below.

D2. On the other hand, the h-principle is wrong for Lagrangian embeddings. As it follows from an unpublished work of H. Hofer and K. Luttinger, any two Lagrangian embeddings $T^2 \to \mathbb{C}^2$ are Lagrangian isotopic.

E. Free maps. A map $T^2 \to \mathbb{R}^n$ is called *free* if 5 vectors

$$\frac{\partial f}{\partial x_1}(x), \frac{\partial f}{\partial x_2}(x), \frac{\partial^2 f}{\partial x_1 \partial x_2}(x), \frac{\partial^2 f}{\partial x_1^2}(x), \frac{\partial^2 f}{\partial x_2^2}(x) \in \mathbb{R}^n$$

are linearly independent for all $x \in T^2$. Of course, the minimal dimension n for which free embeddings may exist is equal to 5.

It is an open problem whether there exists a free map $T^2 \to \mathbb{R}^5$. In particular, we do not know whether the h-principle holds for free maps to $\mathbb{R}^5$. On the other hand, free maps to $\mathbb{R}^6$ satisfy the h-principle. We invite the reader to guess what this statement really means, or look at [**GE71**].

F. Contact and Engel structures. A *contact structure* on a 3-manifold M is a completely non-integrable tangent 2-plane field. A completely non-integrable tangent 2-plane field on a 4-manifold N is called an *Engel structure.* The complete non-integrability means in the first case that the Lie brackets (resp. two successive Lie brackets it the second case) of tangent to the plane field vector fields generate TM at each point of M.

F1. Some forms of the h-principle holds in the contact case even for closed manifolds. For instance, *any tangent to M plane field is homotopic to a contact structure* (see [**Lu77**] and Section 11.2 below).

F2. On the other hand, it is unknown whether the h-principle holds for Engel structures on closed 4-manifolds. In particular, *it is an outstanding open question whether any closed parallelizable* 4*-manifold admits an Engel structure.* ►

Part 1

Holonomic Approximation

Chapter 1

Jets and Holonomy

1.1. Maps and sections.

It is customary to visualize a map $f : \mathbb{R}^n \to \mathbb{R}^q$ as its graph $\Gamma_f \subset \mathbb{R}^n \times \mathbb{R}^q$. This graph may be considered as an image of a map $\mathbb{R}^n \to \mathbb{R}^n \times \mathbb{R}^q$ given by the formula $x \mapsto (x, f(x))$. Mathematicians call this map a *section*, while Physicists prefer to call it a *field* (or an $\mathbb{R}^q$-valued field). Hence any map $f : \mathbb{R}^n \to \mathbb{R}^q$ can be thought of as a section $\mathbb{R}^n \to \mathbb{R}^n \times \mathbb{R}^q$ of the *trivial fibration* $\mathbb{R}^n \times \mathbb{R}^q \to \mathbb{R}^n$. Similarly, any map $V \to W$, where V and W are smooth manifolds, can be considered as a *W-valued field*, or as a section $V \to V \times W$ of the *trivial fibration* $V \times W \to V$. We will also consider arbitrary *fibrations* (=*fiber bundles*) $X \to V$ and *sections* of these fibrations, i.e. maps $f : V \to X$ such that $p \circ f = \mathrm{id}_V$. In all cases the image of a section contains all the information about this section and we will use the term "section" both for the section as a map and for its image.

In what follows we usually denote the dimensions of V, W and X by n, q and $n + q$. By a *section* or a *map* we mean, as a rule, a C^∞-smooth section or a map. By a *family* of sections or maps we mean, as a rule, a *continuous* (with respect to the parameter) family of sections or maps. However such a family is supposed to be *smooth* in the cases when we need to differentiate with respect to the parameter.

1.2. Coordinate definition of jets. The space $J^r(\mathbb{R}^n, \mathbb{R}^q)$

Given a (smooth) map $f : \mathbb{R}^n \to \mathbb{R}^q$, the string of derivatives of f up to the order r

$$J^r_f(x) = \Big(f(x), f'(x), \dots, f^{(r)}(x)\Big)$$

is called the *r-jet* of f at the point $x \in \mathbb{R}^n$. Here $f^{(s)}$ consists of all partial derivatives $D^\alpha f$, $\alpha = (\alpha_1, \dots, \alpha_n)$, $|\alpha| = \alpha_1 + \dots + \alpha_n = s$, written lexicographically.

Let $d_r = d(n, r)$ be the number of all partial derivatives D^α of order r of a function $\mathbb{R}^n \to \mathbb{R}$. The r-jet $J^r_f(x)$ of the map $f : \mathbb{R}^n \to \mathbb{R}^q$ can be considered as a point of the space

$$\mathbb{R}^q \times \mathbb{R}^{qd_1} \times \mathbb{R}^{qd_2} \times \dots \times \mathbb{R}^{qd_r} = \mathbb{R}^{qN_r}$$

where $N_r = 1 + d_1 + d_2 + \dots + d_r$.

◀ **Exercise.** Prove that $d(n, r) = \frac{(n+r-1)!}{(n-1)!r!}$ and $N(n, r) = \frac{(n+r)!}{n!r!}$. ▶

The space $x \times \mathbb{R}^{qN_r}$ can be viewed as a space of all *a priori possible* values of jets of the maps $f : \mathbb{R}^n \to \mathbb{R}^q$ at the point $x \in \mathbb{R}^n$. In this context the space

$$\mathbb{R}^n \times \mathbb{R}^{qN_r} = \mathbb{R}^n \times \mathbb{R}^q \times \mathbb{R}^{qd_1} \times \mathbb{R}^{qd_2} \times \dots \times \mathbb{R}^{qd_r}$$

is called the *space of r-jets* of maps $\mathbb{R}^n \to \mathbb{R}^q$, or the *space of r-jets* of sections $\mathbb{R}^n \to \mathbb{R}^n \times \mathbb{R}^q$, and denoted by $J^r(\mathbb{R}^n, \mathbb{R}^q)$. For example,

$$J^1(\mathbb{R}^n, \mathbb{R}^q) = \mathbb{R}^n \times \mathbb{R}^q \times M_{q\times n}$$

where $M_{q\times n} = \mathbb{R}^{qn}$ is the space of $(q \times n)$-matrices. Given a section $f : \mathbb{R}^n \to \mathbb{R}^n \times \mathbb{R}^q$, the section

$$J^r_f : \mathbb{R}^n \to J^r(\mathbb{R}^n, \mathbb{R}^q)\,, \quad x \mapsto J^r_f(x)\,,$$

of the trivial fibration

$$p^r : J^r(\mathbb{R}^n, \mathbb{R}^q) = \mathbb{R}^n \times \mathbb{R}^{qN_r} \to \mathbb{R}^n$$

is called the *r-jet* of f, or the *r-jet extension* of f.

Note that for any point $z \in J^r(\mathbb{R}^n, \mathbb{R}^q)$ there exists a *unique* $\mathbb{R}^q$-valued polynomial $f(x_1, \dots, x_n)$ of degree r such that $J^r_f(p^r(z)) = z$. Hence, there exists a canonical trivialization

$$\mathbb{R}^n \times P_r(n, q) \overset{J^r}{\mapsto} J^r(\mathbb{R}^n, \mathbb{R}^q)$$

of the fibration $p^r : J^r(\mathbb{R}^n, \mathbb{R}^q) \to \mathbb{R}^n$, where $P_r(n, q)$ is the space of all polynomial maps $\mathbb{R}^n \to \mathbb{R}^q$ of degree $\le r$.

◀ **Exercise.** Draw 1-jets of the function $f : \mathbb{R} \to \mathbb{R};\ \ f(u) = ax + b$. ▶

Figure 1.1. The trivialization φ, the sections f, g, and their images $\varphi_* f$ and $\varphi_* g$.

1.3. Invariant definition of jets

In order to define the space $J^r(V, W)$ of r-jets of sections $V \to V \times W$ of a trivial fibration $p : V \times W \to V$ and, more generally, the r-jet space of sections $V \to X$ of an arbitrary smooth fibration $p : X \to V$, we need to define jets *invariantly.*

Following Gromov's book [**Gr86**] we will use the notation $\mathcal{O}p\, A$ as a replacement of the expression *an open neighborhood of* $A \subset V$. In other words, $\mathcal{O}p\, A$ is an arbitrarily small but non-specified open neighborhood of a subset $A \subset V$.

Let $v \in V$. Two local sections $f : \mathcal{O}p\, v \to X$ and $g : \mathcal{O}p\, v \to X$ of the fibration $X \to V$ are called *r-tangent* at the point v if $f(v) = g(v)$ and

$$J^r_{\varphi_* f}(\varphi(v)) = J^r_{\varphi_* g}(\varphi(v))$$

for a local trivialization $\varphi : U \to \mathbb{R}^n \times \mathbb{R}^q$ of X in a neighborhood U of the point $x = f(v) = g(v)$. Here $\varphi_* f$ and $\varphi_* g$ are images of the sections f and g (see Fig.1.1).

It follows from the chain rule that the r-tangency condition does not depend on the specific choice of the local trivialization. The r-tangency class of a section $f : \mathcal{O}p\, v \to X$ at a point $v \in V$ is called the *r-jet* of f at v and denoted by $J^r_f(v)$. Thus we have correctly defined the *set* $X^{(r)}$ of all r-jets of sections $V \to X$, and the *set-theoretical* fibrations $p^r_0 : X^{(r)} \to X$ and $p^r = p \circ p^r_0 : X^{(r)} \to V$. Moreover, the extensions

$$\varphi^r : (p^r_0)^{-1}(U) \to J^r(\mathbb{R}^n, \mathbb{R}^q)$$

of the local trivializations $\varphi : U \to \mathbb{R}^n \times \mathbb{R}^q$ which send the r-tangency classes of local sections of X to the r-tangency classes of its images in $J^r(\mathbb{R}^n, \mathbb{R}^q)$, define a natural smooth structure on $X^{(r)}$ such that $p^r : X^{(r)} \to V$ becomes a smooth fibration. This fibration is called the *r-jet extension* of the fibration $p : X \to V$. The section

$$J^r_f : V \to X^{(r)}\,,\ v \to J^r_f(v)\,,$$

is called the *r-jet* of a section $f : V \to X$, or the *r-jet extension* of f.

It is important to understand that the chain of inclusions

$$\mathbb{R}^n \times \mathbb{R}^q = J^0(\mathbb{R}^n, \mathbb{R}^q) \subset J^1(\mathbb{R}^n, \mathbb{R}^q) \subset J^2(\mathbb{R}^n, \mathbb{R}^q) \subset \cdots \subset J^r(\mathbb{R}^n, \mathbb{R}^q) \subset \dots$$

is not invariant with respect to fiberwise reparametrizations of $\mathbb{R}^n \times \mathbb{R}^q$. Indeed, the chain rule for the derivatives of order r involves derivatives of all orders $\leq r$. Hence for a general fibration $X \to V$ the chain

$$X = X^{(0)} \subset X^{(1)} \subset X^{(2)} \subset \cdots \subset X^{(r)} \subset \dots$$

does not exist as an invariant object. On the other hand, the r-tangency of two sections implies their s-tangency for all $0 \leq s < r$, and therefore the chain of projections

$$X = X^{(0)} \leftarrow X^{(1)} \leftarrow X^{(2)} \leftarrow \cdots \leftarrow X^{(r)} \leftarrow \dots$$

is invariantly defined.

◀ **Exercise.** Prove that the projection $p^r_{r-1} : X^{(r)} \to X^{(r-1)}$ carries a natural structure of an *affine* bundle. ▶ If X is a trivial fibration $V \times W \to V$ then the space of r-jets of sections (or maps $V \to W$) is denoted by $J^r(V, W)$.

1.4. The space $X^{(1)}$

According to the invariant definition of the jet space, the points of $X^{(1)}$ are classes of 1-tangency of sections, and therefore they can be viewed as non-vertical tangent n-planes $P_x \subset T_x X$. Here “non-vertical” means $P_x \cap \mathrm{Vert}_x = x$, where Vert_x is the q-dimensional tangent space to the fiber of the fibration $X \to V$ over $p(x)$. If we fix a point in $X^{(1)}$, i.e. a non-vertical plane P_x, then the fiber of the fibration $X^{(1)} \to X$ over x can be identified with

$$\mathrm{Hom}(P_x, \mathrm{Vert}_x) \approx \mathrm{Hom}\,(\mathbb{R}^n, \mathbb{R}^q) = \mathbb{R}^{qn}.$$

If $X = V \times W$ and $x = (v, w)$ then $\mathrm{Vert}_x = T_w(v \times W)$, and moreover, we can set $P_x = T_v(V \times w)$. Therefore $J^1(V, W) \to V \times W$ is a vector bundle with the fiber $\mathrm{Hom}\,(T_v V, T_w W)$ over $x = (v, w)$. In particular, $J^1(V, \mathbb{R}) = T^*(V) \times \mathbb{R}$ and $J^1(\mathbb{R}, W) = \mathbb{R} \times T(W)$.

In the general case P_x cannot be canonically chosen, and therefore the fibration $X^{(1)} \to X$ does not have a canonical vector bundle structure, though the affine structure does survive.

Note that the sections of the fibration $p_0^1 : X^{(1)} \to X$ may be identified with connections on X. For example, there exists a natural inclusion $X \to X^{(1)}$ for $X = V \times W$ which corresponds to the standard flat connection on the trivial bundle $V \times W \to V$.

1.5. Holonomic sections of the jet space $X^{(r)}$

Given a section $F : V \to X^{(r)}$ we will denote by $\operatorname{bs} F$ the underlying section $p_0^r \circ F : V \to X$. A section $F : V \to X^{(r)}$ is called *holonomic* if $F = J^r_{\operatorname{bs} F}$. In particular, holonomic sections $\mathbb{R}^n \to J^r(\mathbb{R}^n, \mathbb{R}^q)$ have the form

$$x \mapsto \Big(x, f(x), f'(x), \dots, f^{(r)}(x)\Big) .$$

The correspondence $f \mapsto J^r_f$ defines the derivation map

$$J^r : \operatorname{Sec} X \to \operatorname{Sec} X^{(r)} .$$

Its one-to-one image $J^r(\operatorname{Sec} X)$ coincides with the space $\operatorname{Hol} X^{(r)} \subset \operatorname{Sec} X^{(r)}$ of holonomic sections, i.e. we have

$$\operatorname{Sec} X \overset{J^r}{\simeq} \operatorname{Hol} X^{(r)} \hookrightarrow \operatorname{Sec} X^{(r)} .$$

Note that the C^0-topology on $\operatorname{Sec} X^{(r)}$ induces via J^r the C^r-topology on $\operatorname{Sec} X$.

A homotopy of holonomic sections of $X^{(r)}$ is called a *holonomic homotopy.*

1.6. Geometric representation of sections $V \to X^{(r)}$

A section $F : V \to X^{(1)}$ can be viewed geometrically as a section $f = \operatorname{bs} F : V \to X$ together with a field τ of non-vertical n-planes along f, see Fig.1.2. Such a section is holonomic if and only if the field τ_F is tangent to $f(V)$.

Similarly, a section $F : V \to X^{(s)}$ can be viewed as a pair (F_{s-1}, τ_s) where $F_{s-1} = p^s_{s-1} \circ F : V \to X^{(s-1)}$ and τ_s is a field of non-vertical n-planes along F_{s-1} in $TX^{(s-1)}$. Continuing inductively, we interpret a section $V \to X^{(r)}$ as a the sequence $\{f, \tau_1, \tau_2, \dots, \tau_r\}$ where τ_s is a non-vertical n-plane field in $TX^{(s-1)}$ along $F_{s-1} = \{f, \tau_1, \tau_2, \dots, \tau_{s-1}\}$; $s = 1, \dots, r$. The section $F : V \to X^{(r)}$ is holonomic if and only if for each $s = 1, \dots, r$ the field τ_s is tangent to $F_{s-1}(V)$.

Figure 1.2. 1-jet of a map $\mathbb{R}^n \to \mathbb{R}^q$ as a pair (graph, plane field along graph).

This interpretation of sections $V \to X$ makes it geometrically clear an analytically evident fact that a random section $F : V \to X^{(r)}$ is not holonomic and that the holonomic sections are rather exotic objects in the space $\operatorname{Sec} X^{(r)}$ of all sections of the jet-bundle $X^{(r)}$.

1.7. Holonomic splitting

The following observation shows that given a local holonomic section $F : \{U \subset V\} \to X^{(r)}$, $U \simeq \mathbb{R}^n$, there are plenty of holonomic sections $U \to X^{(r)}$ which are "parallel" to F.

As we already mentioned, the space $J^r(\mathbb{R}^n, \mathbb{R}^q)$ has a tautological parametrization

$$\mathbb{R}^n \times P_r(n,q) \to J^r(\mathbb{R}^n, \mathbb{R}^q)\,, \quad (v, f) \mapsto (v, J^r_f(v))\,,$$

where $P_r(n,q)$ is the space of all polynomial maps $\mathbb{R}^n \to \mathbb{R}^q$ of degree $\leq r$. Such a parametrization has the following nice property, which we call *holonomic trivialization*: *the images of the horizontal fibers* $\mathbb{R}^n \times f \in \mathbb{R}^n \times P_r(n,q)$ *are holonomic sections* J^r_f. In particular,

1.7.1. (Holonomic splitting) *Any holonomic section* $V \to X^{(r)}$ *has a holonomically trivialized tubular neighborhood over any open ball* $U \subset V$.

This observation is a key to the *Thom Transversality Theorem* (see, for example, [**Gr86**]) which we discuss in the next chapter.

Chapter 2

Thom Transversality Theorem

2.1. Generic Properties and Transversality

It is convenient to express the idea of abundance of maps or sections which satisfy a certain property $\mathcal{P}$ by saying that a *generic* map or section satisfies this property. More precise, we say that a *generic section from a space* $\mathcal{S}$ *has a property* $\mathcal{P}$ if the space of maps from $\mathcal{S}$ which has this property is open and everywhere dense in $\mathcal{S}$, or more generally if it can be presented as a *countable intersection* of open and everywhere dense sets. The space of smooth sections of any fibration, and most other functional spaces considered in this book, are so-called, *Baire spaces* which implies, in particular, that sets of generic maps are at least non-empty.

A map $f : V \to W$ is called *transversal* to a submanifold $\Sigma \subset W$ if for each point $x \in V$ one of the following two conditions holds:

- $f(x) \notin \Sigma$ or
- $f(x) \in \Sigma$ and the tangent space $T_{f(x)}W$ is generated by $T_{f(x)}\Sigma$ and $df(T_xV)$.

If $\operatorname{codim}\Sigma > \dim V$ then the second condition can never be satisfied, and thus transversality just means that $f(V) \cap \Sigma = \varnothing$.

The implicit function theorem guarantees that if a map $f : V \to W$ is transversal to Σ then $f^{-1}(\Sigma)$ is a submanifold of V of the same codimension in V as Σ in W.

2.2. Stratified sets and polyhedra

A closed subset S of a manifold V is called *stratified* if it is presented as a union $\bigcup_0^N S_j$ of locally closed submanifolds S_j, called *strata*, such that for each $k = 0, \ldots, N$ we have $\overline{S_k} = \bigcup_{j=k}^N S_j$, where $\overline{S_k}$ is the closure of the stratum S_k. The dimension of a stratified set is the maximal dimension of its strata.

◀ **Examples**

1. Each manifold V with boundary has a stratification with two strata $S_0 = \operatorname{Int} V$ and $S_1 = \partial V$.

2. Given any smooth triangulation of a manifold V, any closed subset which is a union of simplices of the triangulation is stratified by the strata which are interiors of the simplices. We will call stratified sets of this kind *polyhedra*.

3. Any real analytic, or even semi-analytic set (i.e. a set defined by a system of analytic equations and inequalities) can be stratified (see [**GM87**], p. 43). ▶

2.2.1. (The set of matrices of bounded rank) *Let us denote by Σ^i, $i = 0, \ldots, m$, an algebraic subset of the space $M_{q\times n}$ of $q \times n$ matrices, $n \le q$, which consists of matrices of rank $\le n - i$, and by S_i the space of matrices of rank $= n-i$. Then S_i, $i = 0, \ldots, n$ are locally closed submanifolds of $M_{q\times n}$ of codimension $i(q-n+i)$, and $\bigcup_{j=i}^n S_j$ is a natural stratification of Σ^i, i.e. for each $j = 0, \ldots, i$ the stratum S_j consists of smooth points of $\overline{S}^j$.*

Proof. The condition that the rank of a $q \times n$ matrix is precisely equal to $n - i$ is expressed by equating to zero $(q - n + i)i$ minors of order $i + 1$ enveloping a non-zero minor of order $n - i$. It is straightforward to check that this system has maximal rank. □

2.2.2. (Corollary) *Let $\Sigma^i \subset J^1(V, W)$ be the space of 1-jets of maps of*

$$\operatorname{rank} \le \max(n, q) - i\,.$$

Then Σ^i is a stratified subset of codimension $i(|q-n|+i)$.

A map $f : V \to W$ is called *transversal* to a stratified set $\Sigma = \bigcup_0^N S_j \subset W$ if it is transversal to each stratum S_j, $j = 0, \ldots, N$. For a transversal map f the pre-image $f^{-1}(\Sigma)$ of a stratified subset $\Sigma \subset W$ is a stratified subset of V of the same codimension.

2.3. Thom Transversality Theorem

We begin with an almost obvious lemma which is the simplest case of the A. Sard theorem [**Sa42**].

2.3.1. *Let $f : V \to W$ be a (C^1-)smooth map. If $q > n$ then the image $f(V)$ has zero q-dimensional Lebesgue measure.*

Proof. It is sufficient to consider the case when V is the n-disk $D = D^n$ and $W = \mathbb{R}^q$. There is a constant $C > 0$ such that for any integer $N > 0$ the ball D can be covered by CN^n balls D_i of radius $1/N$. Set $\Delta = \max_{a \in D} ||d_a f||$. The image of each ball D_i is contained in a ball $\widetilde{B}_i \subset \mathbb{R}^q$ of radius $\leq \frac{\Delta}{N}$. Hence the total volume of balls $\widetilde{B}_i$ is bounded by

$$\frac{CN^n \Delta^q \sigma_n}{N^q} = \frac{\widetilde{C}}{N^{q-n}} \underset{N \to \infty}{\to} 0 \,,$$

where σ_n is the volume of the unit ball in $\mathbb{R}^n$. Therefore the image $f(V)$ can be covered in W by a set of an arbitrarily small measure. □

2.3.2. (Thom Transversality Theorem) *Let $X \to V$ be a smooth fibration, and Σ a stratified subset of the jet-space $X^{(r)}$. Then for a generic section $f : V \to X$ its jet-extension $J^r_f : V \to X^{(r)}$ is transversal to Σ.*

Proof of Theorem 2.3.2 in case $\mathbf{codim}\,\Sigma > n = \dim V$

We need to prove that for a generic *holonomic* section $F : V \to X^{(r)}$ the image $F(V)$ does not intersect Σ. It is sufficient to consider the case when V is a closed disc D^n. The space of sections

$$\mathrm{Sec}\,(X^{(r)} \setminus \Sigma) \subset \mathrm{Sec}\,X^{(r)}$$

is *open* because $\Sigma \subset X^{(r)}$ is closed and D^n is compact. Hence, the space

$$\mathrm{Hol}\,(X^{(r)} \setminus \Sigma) \subset \mathrm{Hol}\,X^{(r)}$$

is also open. Therefore, we only need to check that $\mathrm{Hol}\,(X^{(r)} \setminus \Sigma)$ is everywhere dense in $\mathrm{Hol}\,X^{(r)}$.

Let $F : D^n \to X^{(r)}$ be a holonomic section. As was explained in Section 1.7 above, the section F has a holonomically trivialized tubular neighborhood $Y = D^n \times \mathbb{R}^K$ where $K = qN_r$. Identifying the vertical fiber

$$0 \times \mathbb{R}^K \subset D^n \times \mathbb{R}^K$$

with the space of horizontal holonomic sections $\{D^n \times a\}_{a \in \mathbb{R}^K}$ one can say that the image $\pi(\Sigma) \subset \mathbb{R}^K$ under the projection $\pi : Y = D^n \times \mathbb{R}^K \to \mathbb{R}^K$ consists of all horizontal sections which are *non-transversal* to Σ. By our assumption $\dim \Sigma < \dim X^{(r)} - n = K$. Hence Lemma 2.3.1 implies that

$\pi(\Sigma)$ has measure 0 in $\mathbb{R}^K$. In particular, the complement $\mathbb{R}^K \setminus \pi(\Sigma)$ is everywhere dense in $\mathbb{R}^K$. Therefore, any open neighborhood of F contains a section $\widetilde{F} \in \mathrm{Hol}\,(X^{(r)} \setminus \Sigma)$. □

Proof of Theorem 2.3.2 in case $\mathbf{codim}\,\boldsymbol{\Sigma} \leq \mathbf{n} = \mathbf{dim}\,\mathbf{V}$

Denote by $\Sigma^{(1)}$ the subset of the jet space $X^{(r+1)}$ which consists of the $(r+1)$-jets of sections $f : \mathcal{O}p\, v \to X$ for which J^r_f is not transversal to Σ. In the case $\operatorname{codim} \Sigma \leq n$ the assertion that a generic section in $\mathrm{Hol}\, X^{(r)}$ is transversal to Σ is equivalent to the assertion that a generic section in $\mathrm{Hol}\, X^{(r+1)}$ belongs to $\mathrm{Hol}\,(X^{(r+1)} \setminus \Sigma^{(1)})$. Hence, this case of 2.3.2 can be deduced from the one considered above and the following lemma.

2.3.3. *Let $\Sigma \subset X^{(r)}$ be a stratified subset. Suppose that $k = \operatorname{codim} \Sigma \leq n = \dim V$. Then $\Sigma^{(1)}$ is a stratified subset of $X^{(r+1)}$ of codimension $n+1$.*

We will illustrate the ideas involved in the proof of Lemma 2.3.3 in a couple of partial cases and will leave the general case as an (advanced) exercise to the reader.

Case $\mathbf{r = 0}$. We may assume that $X \to V$ is the trivial fibration $\mathbb{R}^n \times \mathbb{R}^q \to \mathbb{R}^n$ and Σ is a coordinate subspace in $\mathbb{R}^n \times \mathbb{R}^q$. We denote by $(x_1, \dots, x_n)$ the coordinates in $\mathbb{R}^n$ and by $(y_1, \dots, y_q)$ the coordinates in $\mathbb{R}^q$. Let

$$\Sigma = \{y_1 = \dots = y_k = 0\}\,.$$

Then $X^{(1)} = J^1(\mathbb{R}^n, \mathbb{R}^q) = \mathbb{R}^n \times \mathbb{R}^q \times M_{q\times n}$, where $M_{q\times n}$ is the space of $(q \times n)$-matrices $Z = (z_{ij})$. Let $\widetilde{Z}$ be the $(k \times n)$-submatrix which consists of the first k rows of Z. The set $\Sigma^{(1)}$ is given in this notation by the conditions

$$\begin{cases} y_1 = \dots = y_k = 0 \\ \operatorname{rank} \widetilde{Z} \leq k-1\,. \end{cases}$$

According to 2.2.1 the above equations define a stratified subset of $X^{(1)}$ of codimension $k + (n-k+1) = n+1$. □

Case $\mathbf{codim}\,\boldsymbol{\Sigma} = \mathbf{1}$. We may assume as above that $p : X \to V$ is the trivial fibration $\mathbb{R}^n \times \mathbb{R}^q \to \mathbb{R}^n$. Let us denote the coordinates in the space $X^{(r)}$ by

$$x_1\,, \dots\,, x_n\,, y_1^\alpha\,, \dots\,, y_q^\alpha\,,$$

where $\alpha = (\alpha_1, \dots, \alpha_n)$ are multi-indexes with $\alpha_i \geq 0$ and $|\,\alpha| = \sum_{i=1}^{n} \alpha_i \leq r$. The coordinate y_i^α corresponds to the partial derivative

$$\frac{\partial^{|\,\alpha|} f_i}{\partial x^\alpha} = \frac{\partial^{|\,\alpha|} f_i}{\partial x_1^{\alpha_1} \dots \partial x_n^{\alpha_n}}$$

of the coordinate function f_i , $i = 1, \dots q$, of a section $V \to X$. The projections

$$p^r : X^{(r)} \to V \text{ and } p^r_s : X^{(r)} \to X^{(s)} , \quad s = 0, \dots , r - 1$$

(see 1.3) are given by dropping the coordinates y^α_i with $|\alpha| \geq s + 1$ for $s = -1, 0, \dots r - 1$ and $i = 1, \dots , q$. Without loss of generality we may assume that Σ is a submanifold. Locally near a point $z \in \Sigma$ the submanifold Σ can be defined by an equation $F = 0$. Let us denote by $X^r_s(z)$ and $X^r(z)$ the fibers of the projections p^r_s and p^r through the point z.

Suppose that Σ is not transversal to the fiber $X^r(z)$ at z. Then for $x = p^r(z)$ any local section $J^r_{\tilde f} : \mathcal{O}p\, x \to X^{(r)}$, which is C^1-close to a section $J^r_f : \mathcal{O}p\, x \to X^{(r)}$ with $J^r_f(x) = z$, is transversal to Σ. Hence in this case $\Sigma^{(1)} \subset X^{(r+1)}$ does not intersect a neighborhood of the fiber $X^{r+1}_r(z)$. Suppose now that Σ is transversal to the fiber $X^r(z)$ and set

$$S = \max \{s = 0, \dots r |\ \Sigma \text{ is transversal to the fiber } X^r_{s-1}(z)\} ,$$

where $X^r_{-1} = X^r$. Then

$$\frac{\partial F}{\partial y^\alpha_i}(z) = 0$$

for any $i = 1, \dots , q$ when $|\,\alpha| > S$ and there exist $i' \in \{1, \dots , q\}$ and a multi-index α' with $|\,\alpha'| = S$ such that

$$\frac{\partial F}{\partial \bar y}(z) \neq 0 \quad \text{where } \bar y = y^{\alpha'}_{i'} .$$

The tangent space T_z to a section J^r_f at the point z is generated by the vectors

$$v_k = \left(\frac{\partial}{\partial x_k} , \frac{\partial J^r_f}{\partial x_k} \right) , \quad k = 1, \dots , n ,$$

with coordinates

$$x_l = \delta^k_l , \ l = 1, \dots , n , \ \text{ and } \ y^\alpha_i = \frac{\partial^{|\,\alpha|+1} f_i}{\partial x_k \partial x^\alpha}(z) = \frac{\partial^{|\,\alpha|+1} f_i}{\partial x^{\alpha+\delta^k_l}}(z) , \ i = 1, \dots , q .$$

Therefore the set $\Sigma^{(1)} \subset X^{(r+1)}$ can be defined in a neighborhood of the fiber $X^{r+1}_r(z)$ by the system of $(n + 1)$ equations

$$\begin{cases} F = 0 \\ \frac{\partial F}{\partial x_k} + \sum\limits_{i=1}^{q} \sum\limits_{|\alpha| \leq S} \frac{\partial F}{\partial y^\alpha_i} \, \partial_k y^\alpha_i = 0 , \ k = 1, \dots , n , \end{cases}$$

where we *denote* by $\partial_k y^\alpha_i$ the *coordinate* $y^{\alpha+\delta^k_l}_i$ (once more: here $\partial_k y^\alpha_i$ is a coordinate in the jet space, not a derivative!). We claim that the rank of this system equals $n+1$. Indeed, its minor of order $(n+1)$ which consists of columns corresponding to the derivatives with respect to the *coordinates*

$$\bar{y}\,, \partial_1 \bar{y}\,, \dots\,, \partial_n \bar{y}$$

(where $\bar{y} = y_{i'}^{\alpha'}$) has the form

$$\begin{vmatrix} \frac{\partial F}{\partial \bar{y}} & 0 & 0 & 0 & \dots & 0 \\ * & \frac{\partial F}{\partial \bar{y}} & 0 & 0 & \dots & 0 \\ * & 0 & \frac{\partial F}{\partial \bar{y}} & 0 & \dots & 0 \\ * & 0 & 0 & \frac{\partial F}{\partial \bar{y}} & \dots & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ * & 0 & 0 & 0 & \dots & \frac{\partial F}{\partial \bar{y}} \end{vmatrix} = \left(\frac{\partial F}{\partial \bar{y}} \right)^{n+1} ,$$

and hence it does not vanish near the fiber $X_r^{r+1}(z)$. □

Chapter 3

Holonomic Approximation

The *Holonomic Approximation Theorem* which we discuss in this chapter shows that in some sense there are unexpectedly many holonomic sections *near* any submanifold $A \subset V$ of positive codimension.

3.1. Main theorem

Question: *Is it possible to approximate any section $F : V \to X^{(r)}$ by a holonomic section? In other words, given an r-jet section and an arbitrarily small neighborhood of the image of this section in the jet space, can one find a holonomic section in this neighborhood?*

The answer is evidently negative (excluding, of course, the situation when the initial section is already holonomic). For instance, in the case $r = 1$ and $X = V \times \mathbb{R}$ the question has the following geometric reformulation: given a function and an n-plane field along the graph of this function, can one C^0-perturb this graph to make it almost tangent to the given field?

The problem of finding a holonomic approximation of a section of the r-jet space *near a submanifold* $A \subset \mathbb{R}^n$ is also usually unsolvable. The only exception is the zero-dimensional case: any section can be approximated near any point by the r-jet of the respective Taylor polynomial map.

In contrast, the following theorem says that we *always can find* a holonomic approximation of a section $F : V \to X^{(r)}$ *near a slightly deformed* submanifold $\widetilde{A} \subset V$ if the original submanifold $A \subset V$ is of positive codimension.

3.1.1. (Holonomic Approximation Theorem) *Let $A \subset V$ be a polyhedron of positive codimension and $F : \mathcal{O}p\, A \to X^{(r)}$ be a section. Then for arbitrarily small $\delta, \varepsilon > 0$ there exist a δ-small (in the C^0-sense) diffeotopy $h^\tau : V \to V$, $\tau \in [0,1]$, and a holonomic section $\widetilde{F} : \mathcal{O}p\, h^1(A) \to X^{(r)}$ such that*

$$\mathrm{dist}(\widetilde{F}(v)\,,\, F|_{\mathcal{O}p\, h^1(A)}(v)) < \varepsilon$$

for all $v \in \mathcal{O}p\ h^1(A)$ (see Fig.3.1).

Figure 3.1. The sets A, $h^1(A)$, $\mathcal{O}p\, A$ (gray) and $\mathcal{O}p\ h^1(A)$ (deep gray).

◀ **Remarks**

1. If A (and hence V) is *non-compact* then instead of arbitrarily small *numbers* $\varepsilon, \delta > 0$ one can take arbitrarily small positive *functions* $\delta, \varepsilon : V \to \mathbb{R}_+$. Later in the book the similar situations will appear frequently and we always silently assume that our “arbitrarily small numbers” become “arbitrarily small functions” in the case of a non-compact polyhedron A.

2. Let us recall that we use the notation $\mathcal{O}p\, A$ as a replacement of the expression *an open neighborhood of* A and the term *polyhedron* in the sense that A is a subcomplex of a certain smooth triangulation of the manifold V.

3. We assume that the manifold V is endowed with a Riemannian metric and the bundle $X^{(r)}$ is endowed with an Euclidean structure in a neighborhood U of the section $F(V) \subset X^{(r)}$.

4. A diffeotopy $h^\tau : V \to V$, $\tau \in [0,1]$, is called δ-*small*, if $h^0 = \mathrm{Id}_V$ and $\mathrm{dist}(h^\tau(v), v) < \delta(v)$ for all $v \in V$ and $\tau \in [0,1]$.

5. We assume that the image $h^1(A)$ is contained in the domain of definition of the section F. ▶

As we will see below, the *relative* and the *parametric* versions of the theorem are also true. In the relative version the section F is assumed to be already holonomic over $\mathcal{O}p\, B$, where B is a subpolyhedron of A, while the diffeotopy h^τ is constructed to be fixed on $\mathcal{O}p\, B$ and $\widetilde{F}$ is required to coincide with F on $\mathcal{O}p\, B$. Here is the parametric version of 3.1.1.

3.1.2. (Parametric Holonomic Approximation Theorem) *Let $A \subset V$ be a polyhedron of positive codimension and $F_z : \mathcal{O}p\, A \to X^{(r)}$ be a family of sections parametrized by a cube I^m, $m = 0, 1, \dots$. Suppose that the sections F_z are holonomic for $z \in \mathcal{O}p\, \partial I^m$. Then for arbitrarily small $\delta, \varepsilon > 0$ there exist a family of δ-small diffeotopies $h_z^\tau : V \to V$, $\tau \in [0,1]$, $z \in I^m$, and a family of holonomic sections $\widetilde{F}_z : \mathcal{O}p\, h_z^1(A) \to X^{(r)}$, $z \in I^m$, such that*

- $h_z^\tau = \mathrm{Id}_V$ *and* $\widetilde{F}_z = F_z$ *for all* $z \in \mathcal{O}p\, \partial I^m$;
- $\mathrm{dist}(\widetilde{F}_z(v)\,,\, F_z|_{\mathcal{O}p\, h_z^1(A)}(v)) < \varepsilon$ *for all* $v \in \mathcal{O}p\, h_z^1(A)$.

◀ **Remark.** Note that what we call here and below a *parametric version* is also *relative with respect to a subspace of the space of parameters.* ▶

3.2. Holonomic approximation over a cube

Using induction over the skeleton of the polyhedron A and taking into account that the fibration $X \to V$ is trivial over simplices we reduce the relative version of Theorem 3.1.1 to its special case for the pair $(A, B) = (I^k, \partial I^k) \subset \mathbb{R}^n$.

3.2.1. (Holonomic approximation over a cube) *Let $I^k \subset \mathbb{R}^n$, $k < n$, be the unit cube in the coordinate subspace $\mathbb{R}^k \subset \mathbb{R}^n$ of the first k coordinates. For any section*

$$F : \mathcal{O}p\, I^k \to J^r(\mathbb{R}^n, \mathbb{R}^q)$$

which is holonomic over $\mathcal{O}p\, \partial I^k$ and for an arbitrarily small positive numbers $\delta, \varepsilon > 0$ there exist a δ-small (in the C^0-sense) diffeomorphism

$$h : \mathbb{R}^n \to \mathbb{R}^n\,, \quad h(x_1, \dots, x_n) = (x_1, \dots, x_{n-1}, x_n + \varphi(x_1, \dots, x_n))\,,$$

and a holonomic section

$$\widetilde{F} : \mathcal{O}p\, h^1(I^k) \to J^r(\mathbb{R}^n, \mathbb{R}^q)$$

such that

- $h = \mathrm{Id}$ *and* $\widetilde{F} = F$ *on* $\mathcal{O}p\, \partial I^k$;
- $||\widetilde{F} - F|_{\mathcal{O}p\, h(I^k)}||_{C^0} < \varepsilon$.

Theorem 3.2.1 will be deduced from the Inductive Lemma 3.4.1 which we formulate below. In order to formulate the Inductive Lemma we need the notion of a *fiberwise holonomic section.*

3.3. Fiberwise holonomic sections

Given an arbitrary subset $A \subset V$ a section $F : A \to X^{(r)}$ is called *holonomic* if there exists a holonomic extension $\widetilde{F} : \mathcal{O}p\, A \to X^{(r)}$ such that $\widetilde{F}|_A = F$.

Note that any two holonomic extensions $\mathcal{O}p\, A \to X^{(r)}$ of a section $F : A \to X^{(r)}$ can be joined by a homotopy in the space of holonomic extensions. Moreover, the space of holonomic extensions is contractible.

A section $F : V \to X^{(r)}$ is called *holonomic over* $A \subset V$ if the restriction $F|_A$ is holonomic. Given a fibration $\pi : V \to B$ we say that a section $F : V \to X^{(r)}$ is *fiberwise holonomic* . if there exists a continuous family of holonomic extensions $\widetilde{F}_b : \mathcal{O}p\, \pi^{-1}(b) \to X^{(r)}$, $b \in B$, such that for each $b \in B$ the sections $\widetilde{F}_b$ and F coincide over the fiber $\pi^{-1}(b)$. The continuity of the family of sections $\widetilde{F}_b : \mathcal{O}p\, \pi^{-1}(b) \to X^{(r)}$, $b \in B$, means the continuity of the section $\widetilde{F} : \mathcal{O}p\, \widetilde{V} \subset V \times B \to X^{(r)} \times B$ where $\widetilde{V} = \{(v, \pi(v)), v \in V\}$ is the graph of the projection π, and the restriction of $\widetilde{F}$ to $\mathcal{O}p\, \widetilde{V} \cap V \times b$ coincides with $\widetilde{F}_b$.

3.3.1. *Any section $F : V \to X$ is holonomic over any point $v \in V$. Moreover, it is fiberwise holonomic with respect to the trivial fibration $id_V : V \to V$.*

Indeed, locally we can take the Taylor polynomial map which corresponds to $F(v)$ with respect to some local coordinate system centered at v as a section $\widetilde{F}_v : \mathcal{O}p\, v \to X^{(r)}$. Then the global result follows using a partition of unity and the contractibility of the space of holonomic extensions. □

The contractibility of the space of holonomic extensions also implies:

3.3.2. *Suppose that for closed sets $B \subset A \subset V$ a section $F : \mathcal{O}p\, A \to X^{(r)}$ is holonomic over $\mathcal{O}p\, B$. Then there exists a family of holonomic extensions*

$$\widetilde{F}_b : \mathcal{O}p\, v \to X^{(r)},\ v \in A\,,$$

such that $F_v(v) = F(v)$ for all $v \in A$, and $\widetilde{F}_v = F|_{\mathcal{O}p\, v}$ for $v \in B$.

The above statement also holds parametrically for families of sections.

3.4. Inductive Lemma

In the induction below we will consider the cube $I^k \subset \mathbb{R}^n$ as the family $\{y \times I^l\}_{y \in I^{k-l}}$ of l-dimensional cubes, $l = 0, 1, \dots, k-1$. We recommend to the reader to keep in mind two simplest cases while reading for the first time the statements and the proofs in this and the next sections: $n = 2$, $k = 1$, $l = 0$ and $n = 3$, $k = 2$, $l = 1$. We will illustrate these cases by pictures.

Given a subset $A \subset \mathbb{R}^n$ we will denote by $\mathcal{N}_\delta(A)$ its cubical δ-neighborhood. Let $\pi_s : \mathbb{R}^n \to \mathbb{R}^s$ be the projection to the space of first s coordinates. Let us fix a positive $\theta < 1$ and for $y \in I^{k-l} \subset I^k \subset \mathbb{R}^n$ we set

$$U_\delta(y) = \mathcal{N}_\delta(y \times I^l)\,,\ V_\delta(y) = \mathcal{N}_\delta(y \times \partial I^l)\,,$$

$$A_\delta(y) = \left(\overline{U_{\delta_1}(y)} \setminus V_\delta(y)\right) \cap \pi_{k-l}^{-1}(y), \quad \text{where} \quad \delta_1 = \theta\delta\,,$$

see Fig.3.2 and Fig.3.3.

◀ **Remark.** In all our considerations below in this chapter we can proceed with *any* fixed positive $\theta < 1$. However, for some further generalizations in Section 15.2 it will be convenient to take $\theta \leq \frac{1}{4}$. ▶

Figure 3.2. The sets $U_\delta(y)$ and $A_\delta(y)$, the case $n = 2$, $k = 1$, $l = 0$.

Figure 3.3. The sets $U_\delta(y)$, $V_\delta(y)$ and $A_\delta(y)$, the case $n = 3$, $k = 2$, $l = 1$.

3.4.1. (Inductive Lemma, first version) *Let $I^k \subset \mathbb{R}^n$, $k < n$, be the unit cube in the coordinate subspace $\mathbb{R}^k \subset \mathbb{R}^n$ of the first k coordinates. Suppose that a section*

$$F : \mathcal{O}p\, I^k \to J^r(\mathbb{R}^n, \mathbb{R}^q)$$

is holonomic over $\mathcal{O}p\,\partial I^k$ and for a non-negative integer $l < k$ it is fiberwise holonomic with respect to the fibration $\pi_{k-l} : I^k \to I^{k-l}$, i.e. along the cubes

$$y \times I^l\,, \quad y = (z,t) \in I^{k-l} = I^{k-l-1} \times I\,.$$

More precise, suppose that for a positive δ there exists a family of holonomic sections

$$F_y = J^r_{f_y} : U_\delta(y) \to J^r(U_\delta(y), \mathbb{R}^q)\,, \quad y \in I^{k-l}\,,$$

such that

- $F_y|_{(y\times I^l)\cup V_\delta(y)} = F|_{(y\times I^l)\cup V_\delta(y)}$ *;*
- $F_y = F|_{U_\delta(y)}$ *for* $y \in \mathcal{O}p\,\partial I^{k-l}$ *.*

Then for an arbitrarily small $\varepsilon > 0$ there exist an integer $N > 0$ and a family of holonomic sections

$$\widetilde{F}_z : \Omega_z \to J^r(\mathbb{R}^n, \mathbb{R}^q)\,, \quad z \in I^{k-l-1}\,,$$

where

$$\Omega_z = \mathcal{O}p\left(\bigcup_{i=1}^{N} A_\delta(z, c_i) \cup z \times I^{l+1}\right) \setminus \bigcup_{i=1}^{N} A_\delta(z, c_i)\,,$$

$c_i = \frac{2i-1}{2N}$, $i = 1, \dots, N$, *(see Fig.3.4 and Fig.3.5) such that*

- $\widetilde{F}_z = F$ *on* $\Omega_z \cap \mathcal{O}p\,\partial I^k$ *;*
- $||\widetilde{F}_z - F|_{\Omega_z}||_{C^0} < \varepsilon$ *.*

◀ **Remark.** Note that for $l = k-1$ we have $z \in I^0$ and hence the family Ω_z consists of one domain $\Omega = \Omega_z$. ▶

Figure 3.4. The sets $\bigcup\limits_{i=1}^{N} A_\delta(z, c_i) \cup I^{l+1}$ and Ω (gray) in the case $n = 2$, $k = 1$, $l = 0$.

Figure 3.5. The set $\bigcup_{i=1}^{N} A_\delta(z, c_i) \cup I^{l+1}$, the case $n = 3$, $k = 2$, $l = 1$.

3.4.2. (Inductive Lemma, second version) *Under the conditions of 3.4.1 there exist a δ-small diffeomorphism*

$$h : \mathbb{R}^n \to \mathbb{R}^n\,, \quad h(x_1, \dots, x_n) = (x_1, \dots, x_{n-1}, x_n + \varphi(x_1, \dots, x_n)),$$

and a section

$$\widetilde{F} : \mathcal{O}p\, h(I^k) \to J^r(\mathbb{R}^n, \mathbb{R}^q)$$

such that

- $h = \mathrm{Id}$ *and* $\widetilde{F} = F$ *on* $\mathcal{O}p\, \partial I^k$ *;*
- $||\widetilde{F} - F|_{\mathcal{O}p\, h(I^k)}||_{C^0} < \varepsilon$ *;*
- *the section* $\widetilde{F}|_{h(I^k)}$ *is fiberwise holonomic with respect to the fibration*

$$\pi_{k-l-1} : h(I^k) \to I^{k-l-1}\,,$$

  i.e. along the cubes $h(z \times I^{l+1})$, $z \in I^{k-l-1}$.

◀ **Remark.** In particular, for $l = k - 1$ the new section $\widetilde{F}$ is holonomic as a *whole section.* ▶

3.4.1 ⇒ 3.4.2: There exists a diffeomorphism

$$h : \mathbb{R}^n \to \mathbb{R}^n\,, \quad h(x_1, \dots, x_n) = (x_1, \dots, x_{n-1}, x_n + \varphi(x_1, \dots, x_n))$$

such that $h = \mathrm{Id}$ on $\mathcal{O}p\, \partial I^k$ and for each $z \in I^{k-l-1}$ the image $h(z \times I^{l+1})$ is contained in Ω_z (see Fig.3.6). Then the section $\widetilde{F}_z$ constructed in Lemma 3.4.1 is defined on $\mathcal{O}p\, h(z \times I^{l+1})$, and hence the section

$$\widetilde{F}(z, t, x) = \begin{cases} \widetilde{F}_z(z, t, x)\,, & (z, t, x) \in (\mathcal{O}p\, h(I^k)) \cap (I^{k-l-1} \times \mathbb{R}^{n-k+l+1}) \\ F(z, t, x)\,, & (z, t, x) \in \mathcal{O}p\, (\partial I^k) \end{cases}$$

has the required properties. □

Figure 3.6. The image $h(I)$, the case $n=2$, $k=1$, $l=0$.

3.5. Proof of the Inductive Lemma

We consider first the case $l=k-1$, when F is fiberwise holonomic along the cubes $t\times I^{k-1}\subset I^k$, and thus we need to construct an entirely holonomic section $\widetilde{F}$. Then in the general case we will rewrite the proof almost literally, just incorporating all the way the variable $z\in I^{k-l-1}$ into the notation.

A. The case l=k-1.

In this case $y=t\in I$ and the notation which we introduced at the beginning of this section take the form

$$U_\delta(t)=\mathcal{N}_\delta(t\times I^{k-1})\,,\ V_\delta(t)=\mathcal{N}_\delta(t\times\partial I^{k-1})\,,$$
$$A_\delta(t)=\left(\overline{U_{\delta_1}(t)}\setminus V_\delta(t)\right)\cap\pi_1^{-1}(t)\,.$$

We also set $W_\delta(t)=[U_\delta(t)\setminus\overline{U_{\delta_1}(t)}]\cup V_\delta(t)$.

In what follows δ is fixed and we will write $U(t)$, $A(t)$, $V(t)$ and $W(t)$ instead of $U_\delta(t)$, $A_\delta(t)$, $V_\delta(t)$ and $W_\delta(t)$, respectively.

Set $F_t=F_1$ for $t>1$. Note that

$$\max_{t\in I\,,\,x\in U(t+\sigma)\cap U(t)}||F_{t+\sigma}(x)-F_t(x)||\underset{\sigma\to 0}{\to}0\,,$$

and hence we have the following

3.5.1. (Interpolation Property) *For any $\varepsilon>0$ there exist a number $\sigma=1/N$ and a family of holonomic sections*

$$F_t^\tau:U(t)\to J^r(\mathbb{R}^n,\mathbb{R}^q)\,,\ \ t\in I\,,\,\tau\in[0,\sigma],$$

such that

(a) $F_t^0=F_t$ *for all* $t\in I$*;*

(b) $F_t^\tau|_{W(t)}=F_t|_{W(t)}$ *for all* $t\in I$ *and* $\tau\in[0,\sigma]$ *;*

(c) $||F_t^\tau-F_t||_{C^0}<\varepsilon$ *for all* $t\in I$ *and* $\tau\in[0,\sigma]$*;*

(d) $F_t^\tau|_{\mathcal{O}p\,(t\times I^{k-1})}=F_{t+\tau}|_{\mathcal{O}p\,(t\times I^{k-1})}$ *for all* $t\in I$ *and* $\tau\in[0,\sigma]$ *(see Fig.3.7 and Fig.3.8).*

◀ **Remark.** Note that (d) automatically implies $\sigma < \delta$. In fact, in most cases $\sigma \ll \delta$. ▶

Figure 3.7. The graphs of the sections F_t (schematically), the case $n = 2,\ k = 1,\ l = 0$.

Figure 3.8. The graphs of the sections F_t (left picture) and the sections F_t and F_t^σ (right picture) over $I^k \cap U(t)$ (schematically), the case $n = 2,\ k = 1,\ l = 0$.

For $i = 0, 1, \ldots, N$ set

$$B_i = i\sigma \times I^{k-1}\,.$$

For $i = 1, \ldots, N$ set

$$F_i^{\mathrm{old}} = F_{i\sigma} \qquad F_i^{\mathrm{new}} = F_{i\sigma}^{\sigma}\,,$$

$$c_i = i\sigma - \sigma/2 = \frac{2i-1}{2N}\,,\ \ A_i = A(c_i)\,,\ \ \Delta_i = ((i-1)\sigma, i\sigma)\,,$$

$$\Delta_i^- = ((i-1)\sigma,\ \ c_i],\ \ \Delta_i^+ = [c_i, i\sigma)\,,$$

$$\widetilde{U}_i = U(i\sigma) \cap \pi_1^{-1}(\Delta_i)\,,\ \ \widetilde{U}_i^- = \widetilde{U}_i \cap \pi_1^{-1}(\Delta_i^-)\,,\ \ \widetilde{U}_i^+ = \widetilde{U}_i \cap \pi_1^{-1}(\Delta_i^+)\,,$$

$$\widetilde{U}_i' = \widetilde{U}_i \cap \pi_1^{-1}(c_i) = \widetilde{U}_i^- \cap \widetilde{U}_i^+$$

(see Fig.3.9).

Figure 3.9. The set $W(i\sigma) \subset U(i\sigma)$ (left picture, gray color) and the sets $\widetilde{U}_i$, $\widetilde{U}_i^-$, $\widetilde{U}_i^+$, $\widetilde{U}_i'$, $A(c_i)$ (right picture, gray color), the case $n = 2$, $k = 1$, $l = 0$.

The set $U_i' \setminus A_i$ lies in $W(i\sigma)$ and therefore, according to the Interpolation Property 3.5.1, the section F_i^{new} coincides with F_i^{old} on $U_i' \setminus A_i$. Hence the formula

$$\widetilde{F}(x) = \begin{cases} F_i^{\text{old}}(x), & x \in \widetilde{U}_i^-, \\ F_i^{\text{new}}(x), & x \in \widetilde{U}_i^+, \end{cases}$$

$i = 1, \dots, N$, defines a holonomic section over $\bigcup\limits_{i=1}^{N} (\widetilde{U}_i \setminus A_i)$ (see Fig.3.10). We also have

$$F_{i+1}^{\text{old}} = F_i^{\text{new}}$$

over $\mathcal{O}p\, B_i$ for $i = 1, \dots, N-1$, and hence $\widetilde{F}$ extends continuously to

$$\bigcup_{i=1}^{N} (\widetilde{U}_i \setminus A_i) \cup \bigcup_{0}^{N} \mathcal{O}p\, B_i = \Omega$$

(see Fig.3.11 and Fig.3.12). □

Figure 3.10. The section $\widetilde{F}$ over $\widetilde{U}_i \setminus A_i$, the case $n = 2$, $k = 1$, $l = 0$.

◄ Exercises

1. Prove that for $\sigma > \delta_1$ one can construct an approximating section $\widetilde{F}$ on $\mathcal{O}p\, I^k$.

Figure 3.11. The set $\bigcup\limits_{i=1}^{N}(\widetilde{U}_i \setminus A_i) \cup \bigcup\limits_{0}^{N} \mathcal{O}p\, B_i$ (gray color), the case $n = 2$, $k = 1$, $l = 0$.

Figure 3.12. The section $\widetilde{F}$ over $\bigcup\limits_{i=1}^{N}(\widetilde{U}_i \setminus A_i) \cup \bigcup\limits_{0}^{N} \mathcal{O}p\, B_i$, the case $n = 2$, $k = 1$, $l = 0$.

2. The previous exercise may lead to a (suspicious) conclusion that by choosing a sufficiently small δ_1 one always can construct the approximating section $\widetilde{F}$ over $\mathcal{O}p\, I^k$. Why does this idea fail? ▶

B. The parametric case.

We will proceed parametrically with $z \in I^{k-l-1}$ to produce the families of diffeomorphisms h_z and holonomic sections $\widetilde{F}_z$. We repeat the previous proof almost literally, just systematically inserting the parameter z in our notation.

Recall that for $(z,t)\in I^{k-l-1}\times I$ and $\delta>0$ we have the following notation:

$$U_\delta(z,t)=\mathcal{N}_\delta(z\times t\times I^l)\,,\ V_\delta(z,t)=\mathcal{N}_\delta(\partial(z\times t\times I^l))\,,$$

$$A_\delta(z,t)=\Big(\overline{U_{\delta_1}(z,t)}\setminus V_\delta(z,t)\Big)\cap\pi^{-1}(z,t)\,,$$

and we also set for a fixed positive $\theta<1$

$$W_\delta(z,t)=[U_\delta(z,t)\setminus\overline{U_{\delta_1}(z,t)}]\cup V_\delta(z,t)\ \text{ where }\ \delta_1=\theta\delta\,.$$

As in the non-parametric case we fix δ and write $U(z,t)$, $A(z,t)$, $V(z,t)$ and $W(z,t)$ instead of $U_\delta(z,t)$, $A_\delta(z,t)$, $V_\delta(z,t)$ and $W_\delta(z,t)$ respectively.

Set $F_{z,t}=F_{z,1}$ for $t>1$. Note that

$$\max_{(z,t)\in I^{k-l}\,,\ x\in U(z,t+\sigma)\cap U(z,t)}||F_{z,t+\sigma}(x)-F_{z,t}(x)||\underset{\sigma\to0}{\to}0\,,$$

and hence similarly to 3.5.1 we have

3.5.2. (Parametric Interpolation Property) *For any $\varepsilon>0$ there exist a number $\sigma=1/N$ and a family of holonomic sections*

$$F^\tau_{z,t}:U(z,t)\to J^r(\mathbb{R}^n,\mathbb{R}^q)\,,\ \ (z,t)\in I^{k-l-1}\times I\,,\ \tau\in[0,\sigma]\,,$$

such that

(a) $F^0_{z,t}=F_{z,t}$ *for all* $(z,t)\in I^{k-l-1}\times I$;

(b) $F^\tau_{z,t}|_{W(t)}=F_{z,t}|_{W(t)}$ *for all* $(z,t)\in I^{k-l-1}\times I$ *and* $\tau\in[0,\sigma]$ *;*

(c) $||F^\tau_{z,t}-F_{z,t}||_{C^0}<\varepsilon$ *for all* $(z,t)\in I^{k-l-1}\times I$ *and* $\tau\in[0,\sigma]$*;*

(d) $F^\tau_{z,t}|_{\mathcal{O}p\,(z\times t\times I^{l-1})}=F_{z,t+\tau}|_{\mathcal{O}p\,(z\times t\times I^{l-1})}$ *for all* $(z,t)\in I^{k-l-1}\times I$ *and* $\tau\in[0,\sigma]$ *.*

Similar to the non-parametric case we set for $i=0,1,\dots,N$

$$B_{z,i}=z\times i\sigma\times I^l\,,$$

and for $i=1,\dots,N$ and $z\in I^{k-l-1}$

$$F^{\mathrm{old}}_{z,i}=F_{z,i\sigma}\quad F^{\mathrm{new}}_{z,i}=F^\sigma_{z,i\sigma}\,,$$

$$\widetilde{U}_{z,i}=U(z,i\sigma)\cap\pi^{-1}_{k-l}(z\times\Delta_i)\,,\ \ \widetilde{U}^-_{z,i}=\widetilde{U}_{z,i}\cap\pi^{-1}_{k-l}(z\times\Delta^-_i)\,,$$

$$\widetilde{U}^+_{z,i}=\widetilde{U}_{z,i}\cap\pi^{-1}_{k-l}(z\times\Delta^+_i)\,,$$

$$\widetilde{U}'_{z,i}=\widetilde{U}_{z,i}\cap\pi^{-1}_{k-l}(z,c_i)=\widetilde{U}^-_{z,i}\cap\widetilde{U}^+_{z,i}\,,$$

where we keep using the notation

$$c_i=i\sigma-\frac{\sigma}{2}=\frac{2i-1}{2N}\,,\ \ \Delta^-_i=((i-1)\sigma,c_i]\,,\ \ \Delta^+_i=[c_i,i\sigma)\,.$$

The set $U'_{z,i} \setminus A_{z,i}$ lies in $W(z, i\sigma)$ and therefore, according to the Interpolation Property 3.5.2, the section $F^{\text{new}}_{z,i}$ coincides with $F^{\text{old}}_{z,i}$ on $U'_{z,i} \setminus A_{z,i}$. Hence the formula

$$\widetilde{F}_z(x) = \begin{cases} F^{\text{old}}_{z,i}(x), & x \in \widetilde{U}^-_{z,i}, \\ F^{\text{new}}_{z,i}(x), & x \in \widetilde{U}^+_{z,i}, \end{cases}$$

$i = 1, \dots, N$, defines a family of holonomic sections $\widetilde{F}_z$ over $\bigcup\limits_{i=1}^{N} (\widetilde{U}_{z,i} \setminus A_{z,i})$. We also have

$$F^{\text{old}}_{z,i+1} = F^{\text{new}}_{z,i}$$

over $\mathcal{O}p\, B_{z,i}$ for $i = 0, \dots, N-1$, and hence $\widetilde{F}_z$ extends continuously to

$$\bigcup_{i=1}^{N} (\widetilde{U}_{z,i} \setminus A_{z,i}) \cup \bigcup_{0}^{N} \mathcal{O}p\, B_{z,i} = \Omega_z\,.$$

□

3.6. Holonomic approximation over a cube

We will prove here Theorem 3.2.1 by induction over l. Consider for $l = 0, \dots, k$ the following

Inductive Hypothesis $\mathcal{A}^{(l)}$. *Let $F : \mathcal{O}p\, I^k \to J^r(\mathbb{R}^n, \mathbb{R}^q)$ be a section which is holonomic over $\mathcal{O}p\, \partial I^k$. For arbitrarily small $\delta, \varepsilon > 0$ there exist a δ-small diffeomorphism*

$$h : \mathbb{R}^n \to \mathbb{R}^n\,, \quad h(x_1, \dots, x_n) = (x_1, \dots, x_{n-1}, x_n + \varphi(x_1, \dots, x_n)),$$

and a section

$$\widetilde{F}^l : \mathcal{O}p\, h(I^k) \to J^r(\mathbb{R}^n, \mathbb{R}^q)$$

such that

- $h = \mathrm{Id}$ *and* $\widetilde{F}^l = F$ *on* $\mathcal{O}p\, \partial I^k$*;*
- $||\widetilde{F}^l - F|_{\mathcal{O}p\, h(I^k)}||_{C^0} < \varepsilon$*;*
- *the section $\widetilde{F}^l|_{h(I^k)}$ is fiberwise holonomic with respect to the fibration*
$$\pi_{k-l} : h(I^k) \to I^{k-l}\,,$$
i.e. along the cubes $h(y \times I^l)$, $y \in I^{k-l}$.

Proof of Theorem 3.2.1. Proposition 3.3.2 implies $\mathcal{A}^{(0)}$ with $h = \mathrm{Id}_{\mathbb{R}^n}$ and thus gives us the base for the induction. For $l = 0$ the implication $\mathcal{A}^{(l)} \Rightarrow \mathcal{A}^{(l+1)}$ follows immediately from the Inductive Lemma 3.4.2, but in the general case $l > 0$ we cannot apply 3.4.2 directly because the section $\widetilde{F}^l$ is defined near the deformed cube rather than the original one. Note, however, that the diffeomorphism $h : \mathbb{R}^n \to \mathbb{R}^n$ induces the covering map

$h_* : J^r(\mathbb{R}^n, \mathbb{R}^q) \to J^r(\mathbb{R}^n, \mathbb{R}^q)$. The section $\bar{F}^l = (h_*)^{-1}(\widetilde{F}^l)$ is defined over $\mathcal{O}p\, I^k$, coincides with F near ∂I^k and is fiberwise holonomic with respect to the fibration $\pi_{k-l} : I^k \to I^{k-l}$. Applying Lemma 3.4.1 we can approximate $\bar{F}^l$ by a section $\widetilde{F}'$ over a deformed cube $h'(I^k)$. The section $\widetilde{F}'$ coincides with $\bar{F}^l$ near ∂I^k and fiberwise holonomic with respect to the fibration $\pi_{k-l-1} \circ h' : h'(I^k) \to I^{k-l-1}$. If $\widetilde{F}'$ is sufficiently C^0-close to $\bar{F}^l$, then the section $\widetilde{F}^{l+1} = h_*(\widetilde{F}')$ is the required approximation of F in a neighborhood of $h''(I^k)$, where $h'' = h \circ (h')$. This proves $\mathcal{A}^{(l+1)}$ and Theorem 3.2.1. □

3.7. Parametric case.

It turns out that the Inductive Lemma 3.4.2 implies also the parametric version of Theorem 3.2.1. Namely, we have

3.7.1. (Parametric version of the theorem 3.2.1) *Let F_u, $u \in I^m$, be a family of sections $\mathcal{O}p\, I^k \to J^r(\mathbb{R}^n, \mathbb{R}^q)$ parametrized by the cube I^m. Suppose that $k < n$ and the sections F_z are holonomic over $\mathcal{O}p\, \partial I^k$ for all $u \in I^m$ and holonomic over the whole I^k for $u \in \mathcal{O}p\, \partial I^m$. Then for arbitrarily small $\delta, \varepsilon > 0$ there exist a family of δ-small diffeomorphisms*

$$h_u : \mathbb{R}^n \to \mathbb{R}^n\,, \quad h_u(x_1, \dots, x_n) = (x_1, \dots, x_{n-1}, x_n + \varphi_u(x_1, \dots, x_n)),$$

and a family of holonomic sections

$$\widetilde{F}_u : \mathcal{O}p\, h_u(I^k) \to J^r(\mathbb{R}^n, \mathbb{R}^q)$$

such that

- $h_u = \mathrm{Id}$ *and* $\widetilde{F}_u = F_u$ *on* $\mathcal{O}p\, \partial I^k$;
- $h_u = \mathrm{Id}$ *and* $\widetilde{F}_u = F_u$ *for* $u \in \mathcal{O}p\, \partial I^m$;
- $||\widetilde{F}_u - F_u|_{\mathcal{O}p\, h_u(I^k)}||_{C^0} < \varepsilon$.

Proof. Consider the cube $I^{m+k} = I^m \times I^k \subset \mathbb{R}^m \times \mathbb{R}^n = \mathbb{R}^{m+n}$. Let $J^r(\mathbb{R}^{m+n}|\mathbb{R}^n, \mathbb{R}^q)$ be the bundle over $\mathbb{R}^m \times \mathbb{R}^n$ whose restriction to $u \times \mathbb{R}^n$, $u \in \mathbb{R}^m$, equals $J^r(\mathbb{R}^n, \mathbb{R}^q)$. The family of sections

$$F_u : I^k \to J^r(\mathbb{R}^n, \mathbb{R}^q)$$

can be viewed as a section

$$\overline{F} : I^{m+k} \to J^r(\mathbb{R}^{m+n}|\mathbb{R}^n, \mathbb{R}^q)\,.$$

The section $\overline{F}$ lifts to a section $\overline{\overline{F}} : I^{m+k} \to J^r(\mathbb{R}^{m+n}, \mathbb{R}^q)$, so that $\pi \circ \overline{\overline{F}} = \overline{F}$, where

$$\pi : J^r(\mathbb{R}^{m+n}, \mathbb{R}^q) \to J^r(\mathbb{R}^{m+n}|\mathbb{R}^n, \mathbb{R}^q)$$

is the canonical projection. Moreover, the section $\overline{\overline{F}}$ can be chosen holonomic near ∂I^{m+k}. Hence we can apply Theorem 3.2.1 to get an ε-approximation $\widetilde{\widetilde{F}}$ of $\overline{\overline{F}}$ over a δ-displaced cube $h(I^{m+k})$. Then the composition $\widetilde{F} = \pi \circ$

$\widetilde{\widetilde{F}} : I^{m+k} \to J^r(\mathbb{R}^{n+m}|\mathbb{R}^n, \mathbb{R}^q)$ can be interpreted as the required family $\{\widetilde{F}_u\}_{u\in I^m}$ of holonomic ε-approximations of the family $\{F_u\}$ near $\{h_u(I^k)\}$. $\square$

In the same way as Theorem 3.2.1 implies the Holonomic Approximation Theorem, i.e. by induction over skeleta, Theorem 3.7.1 implies the Parametric Holonomic Approximation Theorem 3.1.2.

Chapter 4

Applications

The first two examples below illustrate *Gromov's homotopy principle for open* Diff V*-invariant differential relations over open manifolds* which we formulate and prove later in Part 2 (see 7.2).

4.1. Functions without critical points

Let V be the annulus $\delta^2 \le x_1^2 + x_2^2 \le 4$ in $\mathbb{R}^2$.

4.1.1. *There exists a family of functions* $f_t : V \to \mathbb{R}$, $t \in [0,1]$, *such that* $\operatorname{grad} f_t \ne 0$, $f_0(x_1, x_2) = -x_1^2 - x_2^2$ *and* $f_1(x_1, x_2) = x_1^2 + x_2^2$ *(see Fig.4.1).*

Figure 4.1. The functions f_0 and f_1.

Proof. The 1-jet space $J^1(V, \mathbb{R})$ equals $V \times \mathbb{R} \times \mathbb{R}^2$ and we will identify the last factor, which is reserved for the gradient of a function, with the complex line $\mathbb{C}$. Note that $\operatorname{grad} f_0 = -\operatorname{grad} f_1$. The family of sections of $J^1(V, \mathbb{R})$ defined by the formula

$$F_t = ((1-t)f_0 + t f_1, \quad e^{i\pi t} \operatorname{grad} f_0)$$

joins $F_0 = J^1_{f_0}$ with $F_1 = J^1_{f_1}$. For $t \neq 0, 1$ the section F_t is not holonomic. We can reparametrize the family F_t making it independent of t, and thus holonomic for $t \in \mathcal{O}p\, \partial I$. Applying the Parametric Holonomic Approximation Theorem 3.1.2 with $A = S^1 \subset V$ one can construct a family of holonomic ε-approximations $\widetilde{F}_t = J^1_{\widetilde{f}_t} : U_t \to J^1(V, \mathbb{R})$ where U_t is a neighborhood of a perturbed circle $h^1_t(S^1)$. Moreover, one can choose $\widetilde{F}_t$ and U_t such that $U_t = V$ and $\widetilde{F}_t = F_t$ for $t \in \mathcal{O}p\ \partial I$. For sufficiently small ε the functions $\widetilde{f}_t$ do not have critical points on U_t because

$$\operatorname{grad} \widetilde{f}_t \approx e^{i\pi t} \operatorname{grad} f_0 \neq 0 \text{ near } S^1.$$

Let $\{\varphi^\tau_t : V \to V,\ \tau \in [0, 1]\}_{t \in [0,1]}$ be a family of isotopies such that for each $t \in [0, 1]$ the isotopy φ^τ_t, $\tau \in [0, 1]$, shrinks V into the neighborhood U_t and $\varphi^\tau_0 = \varphi^\tau_1 = \mathrm{Id}_V$. Then the family $g_t = \widetilde{f}_t \circ \varphi^1_t$ consists of functions without critical points on V and interpolates between f_0 and f_1.

◀ **Exercise.** Try to construct the required family explicitly. ▶

4.2. Smale's sphere eversion

Let $\dim V \leq \dim W$. A map $f : V \to W$ is called *immersion* if $\operatorname{rank} f = \dim V$ everywhere on V. If $\dim V = \dim W$ then an immersion $V \to W$ is the same as a *locally diffeomorphic* map. Two immersions are called *regularly homotopic* if they can be connected by a family of immersions.

Denote by V the thickened sphere

$$(1-\delta)^2 \leq x_1^2 + x_2^2 + x_3^2 \leq (1+\delta)^2$$

in $\mathbb{R}^3$. Let

$$\mathrm{inv} : \mathbb{R}^3 \setminus 0 \to \mathbb{R}^3 \setminus 0,\ \ \mathrm{inv}(x) = x/||x||^2\,,$$

be the inversion,

$$\mathrm{r} : \mathbb{R}^3 \to \mathbb{R}^3,\ \ r(x_1, x_2, x_3) = (x_1, x_2, -x_3)\,,$$

the reflection and $\mathrm{i}_V : V \hookrightarrow \mathbb{R}^3$ the inclusion.

4.2.1. (Smale's sphere eversion, [Sm58]) *The map*

$$\mathrm{r} \circ \mathrm{inv} \circ \mathrm{i}_V : V \to \mathbb{R}^3\,,$$

which inverts V (inside out), is regularly homotopic to the inclusion $\mathrm{i}_V : V \to \mathbb{R}^3$.

◀ **Remarks**

1. This counter-intuitive statement is a corollary of S. Smale's celebrated theorem [**Sm58**]. Equivalently it can be formulated by saying that the 2-sphere in $\mathbb{R}^3$ can be turned inside out via a regular homotopy, i.e. via a family of smooth, but possibly self-intersecting surfaces. One can follow the proof below to actually construct this eversion. However, there are much more efficient ways to do that. The explicit process of this eversion became the subject of numerous publications, videos and computer programs.

2. The map $\mathrm{inv} \circ \mathrm{i}_V : V \to \mathbb{R}^3$, which also inverts V (inside out), is not regularly homotopic to the inclusion $\mathrm{i}_V : V \to \mathbb{R}^3$ because these maps induce on V the opposite orientations. ▶

Proof. Let $f_0 = \mathrm{i}_V$ and $f_1 = r \circ \mathrm{inv} \circ \mathrm{i}_V$. Both df_0 and df_1 have rank equal 3 and induce the same orientation on V. Hence the sections

$$df_0, df_1 : V \to J^1(V, \mathbb{R}^3) = V \times \mathbb{R}^3 \times M_{3\times 3}$$

can be viewed as maps

$$V \to \mathbb{R}^3 \times \mathrm{SO}(3)\,.$$

These maps are homotopic because $\pi_2(\mathrm{SO}(3)) = 0$. [1] Let F_t be the the homotopy connecting $F = df_0$ and $F_1 = df_1$. The deformation F_t can be assumed holonomic for t near ∂I. Applying Theorem 3.1.2 with $A = S^2$ one can construct a family of holonomic ε-approximations $\widetilde{F}_t = J^1_{\widetilde{f}_t} : U_t \to J^1(V, \mathbb{R}^3)$, where U_t is a neighborhood of a perturbed sphere $h^1_t(S^2)$. Moreover, one can choose $\widetilde{F}$ and U_t such that $U_t = V$ and $\widetilde{F}_t = F_t$ for $t \in \mathcal{O}p\, \partial I$. If ε is sufficiently small then $\widetilde{f}_t$ is a regular homotopy. As in the previous example, we can compose $\widetilde{f}_t$ with a family of contractions of V into the neighborhoods U_t and get the desired regular homotopy $g_t : A \to \mathbb{R}^3$ which connects f_0 and f_1. □

◀ **Exercise (S.Smale, [Sm58]).** Prove that every immersion $S^2 \to \mathbb{R}^3$ is regularly homotopic to the standard embedding $S^2 \hookrightarrow \mathbb{R}^3$. ▶

4.3. Open manifolds

For further applications we need some information about open manifolds.

A manifold V is called *open* if there are no closed manifolds among its connected components. In particular, any path-connected manifold V with non-empty boundary is open in this sense.

We say that a path $p : [0, \infty) \to V$ connects $v = p(0)$ with ∞, if p is a proper path and $\lim\limits_{t\to\infty} p(t) \in \partial V$ or does not exist.

The following is well known:

[1] In fact, the homotopy between df_0 and df_1 can be easily constructed explicitly without referring to the computation of $\pi_2(\mathrm{SO}(3))$.

Figure 4.2. The paths connecting barycenters with infinity.

Figure 4.3. A deformation which brings V into an arbitrarily small neighborhood of $(n-1)$-skeleton of a triangulation.

4.3.1. *If V is open, then there exists a polyhedron $K \subset V$, $\operatorname{codim} K \geq 1$, such that V can be compressed by an isotopy $\varphi_t : V \to V$, $t \in [0,1]$, into an arbitrarily small neighborhood U of K.*

Proof. Fix a triangulation of V and some (disjoint) paths $[0, \infty) \to V$, which connect all the barycenters of the n-simplices with ∞, see Fig.4.2. Using these paths we can deform V via an isotopy into the complement of the set of barycenters of the n-simplices, and after that into an arbitrarily small neighborhood of the $(n-1)$-skeleton K, see Fig.4.3. Note that in general the image of V does not coincide with a "regular" neighborhood of K. □

Given an open manifold V a polyhedron $V_0 \subset V$ is called a *core* of V if for an arbitrarily small neighborhood U of V_0 there exists a *fixed on* V_0 isotopy $\varphi_t : V \to V$ which brings V to U. Note that core always exists: one can take a subcomplex $K \subset V$ as in 4.3.1 and remove small open neighborhoods of all intersection points $p_i(\mathbb{R}_+) \cap K$, where the p_i are paths which connect barycenters of n-simplices with ∞.

4.4. Approximate integration of tangential homotopies

Let $\pi : \mathrm{Gr}_n W \to W$ be the Grassmanian bundle of n-planes tangent to a q-dimensional manifold W, $q > n$, and V a n-dimensional manifold. Given a *monomorphism* (= fiberwise injective homomorphism) $F : TV \to TW$, we will denote by $\mathrm{G}F$ the corresponding map $V \to \mathrm{Gr}_n W$. Thus the tangential (Gauss) map associated with an immersion $f : V \to W$ can be written as $\mathrm{G}df$.

In what follows we assume that $V \subset W$ is an embedded submanifold and denote by f_0 the inclusion $\mathrm{in}_V : V \hookrightarrow W$. We also assume that the manifolds W and $\mathrm{Gr}_n W$ are endowed with Riemannian metrics.

A homotopy $G_t : V \to \mathrm{Gr}_n W$ such that $G_0 = \mathrm{G}df_0$ and $\pi \circ G_t = f_0$ is called *tangential homotopy* of the inclusion f_0 .

4.4.1. (Approximate integration of tangential homotopies) *Let $K \subset V$ be a polyhedron of positive codimension and $G_t : V \to \mathrm{Gr}_n W$ a tangential homotopy. Then one can approximate G_t near K by an isotopy of embeddings in the following sense: for arbitrarily small $\delta, \varepsilon > 0$ there exist a δ-small diffeotopy $h^\tau : V \to V\,, t \in I$, and an isotopy of embeddings*

$$\widetilde{f}_t : \mathcal{O}p\,_V \widetilde{K} \to W\,, \quad \text{where } \widetilde{K} = h^1(K) \text{ and } \widetilde{f}_0 = f_0|_{\mathcal{O}p\,_V \widetilde{K}}\,,$$

such that the homotopy $\mathrm{G}d\widetilde{f}_t : \mathcal{O}p\,_V \widetilde{K} \to \mathrm{Gr}_n W$ is ε-close to the tangential homotopy $G_t|_{\mathcal{O}p\,_V \widetilde{K}}$.

◀ **Remark.** The relative and the parametric versions of Theorem 4.4.1 are also true. ▶

Proof. Note that the homotopy G_t can be extended in a homotopically canonical way to a family of maps

$$\bar{G}_t : \mathcal{O}p\,_W V \to \mathrm{Gr}_n W\,.$$

For $\bar{\varepsilon} \in (0, \pi/4)$ choose an integer N such that for each interval

$$\Delta_i = [(i-1)/N, i/N]$$

the homotopy $\{\bar{G}_t\}_{t \in \Delta_i}$ is $\bar{\varepsilon}$*-small* in the following sense: the angle between $\bar{G}_{t_1}(w)$ and $\bar{G}_{t_2}(w)$ is less then $\bar{\varepsilon}$ for all $w \in \mathcal{O}p\,_W V$ and $t_1, t_2 \in \Delta_i$. Set $K_0 = K$ and $V_0 = \mathcal{O}p\,_V K_0$. We will proceed with the following steps:

Step 1. The homotopy $\{G_t = \bar{G}_t|_{V_0}\}_{t \in \Delta_1}$ defines a homotopy of sections

$$\{F_t : V_0 \to X_0^{(1)}\,,\ \mathrm{bs}\, F_t = \mathrm{Id}_{V_0}\}_{t \in \Delta_1}$$

of the 1-jet space $X_0^{(1)}$ of the fibration $\pi_0 : X_0 \to V_0$, where $X_0 \subset \mathcal{O}p\,_W V_0$ is a (normal) tubular neighborhood of $V_0 \subset W$. Using Theorem 3.1.1 one can construct a holonomic ε_1-approximation (where ε_1 will be chosen at Step 2) $\widetilde{F}_{1/N}$ of $F_{1/N}$ over $\mathcal{O}p\,h_1^1(K_0)$, where $\{h_1^\tau : V_0 \to V_0\}_{\tau\in I}$ is a δ/N-small diffeotopy. Set

$$K_1 = \widetilde{F}_{1/N}(h_1^1(K_0)) \text{ and } V_1 = \widetilde{F}_{1/N}(\mathcal{O}p\,_{V_0} h_1^1(K_0))\,.$$

Step 2. If the number ε_1 from Step 1 is chosen sufficiently small then the homotopy $\{\bar{G}_t|_{V_1}\}_{t\in\Delta_2}$ defines a homotopy of sections

$$\{F_t : V_1 \to X_1^{(1)}\,,\ \mathrm{bs}\,F_t = \mathrm{Id}_{V_1}\}_{t\in\Delta_2}$$

of the 1-jet space $X_1^{(1)}$ of the fibration $\pi_1 : X_1 \to V_1$, where $X_1 \subset \mathcal{O}p\,_W V_1$ is a (normal) tubular neighborhood of $V_1 \subset W$. Using Theorem 3.1.1 one can construct a holonomic ε_2-approximation $\widetilde{F}_{2/N}$ of $F_{2/N}$ over $\mathcal{O}p\ h_2^1(K_1)$, where $\{h_2^\tau : V_1 \to V_1\}_{\tau\in I}$ is a δ/N-small diffeotopy. Set

$$K_2 = \widetilde{F}_{2/N}(h_1^2(K_1)) \text{ and } V_2 = \widetilde{F}_{2/N}(\mathcal{O}p\,_{V_1} h_2^1(K_1))\,.$$

We can continue this way until $i = N$. Set

$$\widetilde{K} = \pi_0 \circ \cdots \circ \pi_{N-1}(K_N))\,.$$

(see Fig.4.4).

Figure 4.4. The sets K_0, V_0, $\pi_0(K_1)$, $\pi_0(V_1)$ and $\pi_0 \circ \pi_1(K_2)$.

As follows from the construction, there exists a δ-small diffeotopy $h^\tau : V \to V$ such that $\widetilde{K} = h^1(K)$. Set

$$\widetilde{f}_{i/N} = \widetilde{F}_{i/N} \circ \cdots \circ \widetilde{F}_{1/N} \circ f_0|_{\mathcal{O}p\,\widetilde{K}}\,, \quad i = 0, \dots, N\,.$$

Define $\widetilde{f}_t$ for all other $t \in \Delta_i$ as the linear homotopy which connects $\widetilde{f}_{(i-1)/N}$ and $\widetilde{f}_{i/N}$ in X_{i-1}. Thus $\widetilde{f}_t : \mathcal{O}p\,\widetilde{K} \to W$ is an isotopy of embeddings. For sufficiently small $\bar{\varepsilon}$ and ε_i, $i = 1, \dots, n$, the isotopy $\widetilde{f}_t$ satisfies the required approximation property. □

Let $\mathrm{Gr}_n\mathbb{R}^q_0$ be the Grassmanian manifold of n-dimensional *linear* subspaces in $\mathbb{R}^q$. Let us recall that there exists a canonical Euclidean connection $\nabla^\Gamma = P \circ \nabla$ on the tautological n-dimensional fibration $\Gamma \to \mathrm{Gr}_n\mathbb{R}^q_0$, where

$$E : \Gamma \to S = (\mathrm{Gr}_n\mathbb{R}^q_0) \times \mathbb{R}^q$$

is the canonical embedding, $P : S \to E$ the orthogonal projection and ∇ the standard connection on S. This canonical construction is applicable also to $\mathrm{Gr}_n W$ and hence any tangential homotopy G_t can be canonically covered by an isotopy of *fiberwise isometric monomorphisms* $F_t : TV \to TW$. Note that the covering homotopy F_t can be constructed as the limit $F_t = \lim_{N\to\infty} F^N$ where F^N is a homotopy of monomorphisms such that $\mathrm{G}F^N_{i/N} = G_{i/N}$, and for $t \in [(i-1)/N, i/N]$ the homotopy F^N_t is generated by the orthogonal projection $G_{(i-1)/N} \to G_{i/N}$.

◀ **Exercise.** Prove that one can approximate the homotopy F_t near $K \subset V$ by an "almost isometric" isotopy $\widetilde{f}_t : \mathcal{O}p\,_V K \to W$. ▶

4.5. Directed embeddings of open manifolds

Let $A \subset \mathrm{Gr}_n W$ be an arbitrary subset. An immersion $f : V \to W$ is called *A-directed* if $\mathrm{G}df$ sends V into A. If V is an *oriented* manifold then we can also consider A-directed immersions where A is an arbitrary subset in the Grassmanian $\widetilde{\mathrm{Gr}}_n W$ of *oriented* tangent n-planes to a q-dimensional manifold W. Note that by an *embedding* of open manifold we always assume an embedding onto a locally closed submanifold of the target manifold.

For A-directed *embeddings* Gromov proved in [**Gr86**] via his convex integration technique the following theorem

4.5.1. (A-directed embeddings of open manifolds) *If $A \subset \mathrm{Gr}_n W$ is an open subset and $f_0 : V \hookrightarrow W$ is an embedding whose tangential lift $G_0 = \mathrm{G}df_0 : V \to \mathrm{Gr}_n W$ is homotopic to a map $G_1 : V \to A \subset \mathrm{Gr}_n W$, then f_0 can be isotoped to a A-directed embedding $f_1 : V \to W$. Moreover, given a core $K \subset V$ of the manifold V, the isotopy f_t can be chosen arbitrarily C^0-close to f_0 on $\mathcal{O}p\,K$.*

◀ **Remarks**

1. Gromov's proof is discussed in detail in [**Sp00**]. C. Rourke and B. Sanderson gave two independent proofs of this theorem in [**RS97**] and [**RS00**].

2. The parametric version of Theorem 4.5.1 is also true. The relative version for (V, V_0) is false in general, but is true if each connected component of $V \setminus V_0$ has an exit to ∞, i.e. when $\operatorname{Int} V \setminus \mathcal{O}p\, V_0$ has no compact connected components. ▶

Proof. Theorem 4.5.1 follows almost immediately from 4.4.1. Indeed, let $K \subset V$ be a core of V, i.e. a codimension ≥ 1 subcomplex in V such that V can be compressed into an arbitrarily small neighborhood of K by an isotopy fixed on K. Using Theorem 4.4.1 we can approximate G_t near $\widetilde{K} = h^1(K)$ by an isotopy $\widetilde{f}_t : \mathcal{O}p_V \widetilde{K} \to W$. For a sufficiently close approximation the image $\mathrm{G}\widetilde{f}_1(\mathcal{O}p_V \widetilde{K})$ belongs to A. In order to construct the required isotopy f_t we first compress V into $\mathcal{O}p_V h^1(K)$ and then apply $\widetilde{f}_t$. □

◀ **Remark.** The theorem is valid also in the case when the homotopy G_t covers an arbitrary isotopy $g_t : V \to W$, instead of the constant isotopy $g_t \equiv f_0$. Indeed, one can apply the previous proof to the pull-back homotopy $(d\hat{g}_t)^{-1} \circ G_t$ and the pull-back set $\widetilde{A} = \hat{g}_1^*(A)$, where $\hat{g}_t : W \to W$ is a diffeotopy which extends the isotopy $g_t : V \to W$ underlying G_t. ▶

The following version of Theorem 4.5.1 will be useful for applications which we consider in Section 12.1 below.

4.5.2. *Let $A \subset \mathrm{Gr}_n W$ be an open subset, V an open manifold and $f_0 : V \to W$ an embedding whose differential $F_0 = df_0$ is homotopic via a homotopy of monomorphisms $F_t : TV \to TW$, $\operatorname{bs} F_t = f_0$, to a map F_1 with $\mathrm{G}F_1(V) \subset A$. Then f_0 can be deformed by an isotopy $f_t : V \to W$ to a A-directed embedding $f_1 : V \to W$ such that F_1 is homotopic to df_1 through a homotopy of monomorphisms $\widetilde{F}_t : TV \to TW$, $\operatorname{bs} F_t = f_t$, with $\mathrm{G}\widetilde{F}_t(V) \subset A$ for all $t \in I$.*

Proof. Let $\widetilde{f}_t : \mathcal{O}p_V \widetilde{K} \to W$ be the isotopy constructed in 4.4.1. It is sufficient to construct a homotopy Ψ_t between $F_1|_{\mathcal{O}p_V \widetilde{K}}$ and $d\widetilde{f}_1$ such that $\mathrm{G}\Psi_t(\mathcal{O}p_V \widetilde{K}) \subset A$ for all t. The existence of the *underlying* tangential homotopy $\widetilde{G}_t = \mathrm{G}\Psi_t$ follows from the C^0-closeness of the *maps* $\mathrm{G}F_1|_{\mathcal{O}p_V \widetilde{K}}$ and $\mathrm{G}d\widetilde{f}_1$, while the existence of the *covering* homotopy Ψ_t for $\widetilde{G}_t$ follows from the C^0-closeness of the *homotopies* $\mathrm{G}F_t|_{\mathcal{O}p_V \widetilde{K}}$ and $\mathrm{G}d\widetilde{f}_t$. □

4.6. Directed embeddings of closed manifolds

For some special sets $A \subset \mathrm{Gr}_n W$ Theorem 4.5.1 implies a theorem about A-directed embeddings of *closed* manifolds. Let us give here some necessary definitions.

Let $n < m \leq q$. An open set $A \subset \mathrm{Gr}_n W$ is called *m-complete* if there exists an open set $\widehat{A} \subset \mathrm{Gr}_m W$ such that $A = \bigcup_{\widehat{L} \in \widehat{A}} \mathrm{Gr}_n \widehat{L}$.

◀ **Example.** Suppose that $n < k < q$. Let the set $A_0 \subset \mathrm{Gr}_n \mathbb{R}^q_0$ consist of n-planes intersecting trivially the subspace $L = 0 \times \mathbb{R}^{q-k} \subset \mathbb{R}^q$ and let

$$A = \mathbb{R}^q \times A_0 \subset \mathbb{R}^q \times \mathrm{Gr}_n \mathbb{R}^q_0 = \mathrm{Gr}_n \mathbb{R}^q .$$

Then A is k-complete: $\widehat{A} = \mathbb{R}^q \times \widehat{A}_0$ where $\widehat{A}_0$ consists of all k-planes $\eta \subset \mathbb{R}^q$ such that $L \cap \eta = \{0\}$. ▶

4.6.1. (A-directed embeddings of closed manifolds) *Let $A \subset \mathrm{Gr}_n W$ be an open set which is m-complete for some m, $n < m < q$. Then the statement of Theorem 4.5.1 holds for any* **closed** *n-dimensional manifold V.*

Proof. Let us fix the notation. Denote by $\mathrm{Gr}_{m,n} W$ the manifold of all (m,n)-flags on W, where each flag is a pair of tangent planes (L^m, L^n) in $T_w W$ such that $L^n \subset L^m$. Denote by $\widehat{\pi}$ and π the projections $\mathrm{Gr}_{m,n} W \to \mathrm{Gr}_m W$ and $\mathrm{Gr}_{n,m} W \to \mathrm{Gr}_n W$. Set

$$\bar{A} = \{(\widehat{L}, L) \,|\, \widehat{L} \in \widehat{A} \subset \mathrm{Gr}_m W,\ L \in \mathrm{Gr}_n \widehat{L}\,\} \subset \mathrm{Gr}_{n,m} W,$$

where $\widehat{A}$ is the set implied by the definition of m-completeness. Note that $\widehat{\pi}(\bar{A}) = \widehat{A}$ and $\pi(\bar{A}) = A$.

Let $G_t : V \to \mathrm{Gr}_n W$ be the homotopy between the tangential lift $G_0 = \mathrm{G}df_0$ of the embedding f_0 and the map $G_1 : V \to A$. Suppose that the map

$$G_1 : V \to A \subset \mathrm{Gr}_n W$$

lifts to a map

$$\bar{G}_1 : V \to \bar{A} \subset \mathrm{Gr}_{m,n} W .$$

Then the homotopy G_t lifts to a homotopy $\bar{G}_t : V \to \mathrm{Gr}_{m,n} W$, $t \in [0,1]$. We have $G_t = \pi \circ \bar{G}_t$. Set $\widehat{G}_t = \widehat{\pi} \circ \bar{G}_t$, $t \in [0,1]$. Let N be the total space of the vector bundle over V whose fiber over a point $v \in V$ is the normal space to $G_1(v)$ in $\widehat{G}_1(v)$. The embedding f_0 can be extended to an embedding $\widehat{f}_0 : \mathcal{O}p_N V \to W$ such that the tangential lift $\mathrm{G}d\widehat{f}_0$ coincides with $\widehat{G}_0$ over V. Hence we can apply Theorem 4.5.1 to construct an isotopy $\widehat{f}_t : \mathcal{O}p_N V \to W$, such that $\widehat{f}_1$ is an $\widehat{A}$-directed embedding. Then the restriction $f_t = \widehat{f}_t|_V$ is an isotopy between f_0 and an A-directed embedding $f_1 : V \to W$.

In general we cannot guarantee the existence of the global map $\bar{G}_1 : V \to \bar{A}$ which covers the map $G_1 : V \to A$. However, one can avoid this problem using the following localization trick. Theorem 4.5.1 allows us to construct the required isotopy f_t in a neighborhood $\mathcal{O}p\,K$ of the $(n-1)$-skeleton of a triangulation of V. Moreover, the proof of 4.5.1 provides an isotopy whose tangential lift $\mathrm{G}df_t$ is C^0-close to $G_t|_{\mathcal{O}p\,K}$. Hence, one can assume from the very beginning that our original homotopy G_t is constant over $\mathcal{O}p\,K$ and we need to construct a required isotopy f_t on each top-dimensional simplex Δ of the triangulation keeping f_t constant on $\mathcal{O}p\,\partial\Delta$. If the simplices of the triangulation of V are sufficiently small then the map $G_1 : \Delta \to A \subset \mathrm{Gr}_n W$ lifts to a map $\bar{G}_0 : \Delta \to \bar{A} \subset \mathrm{Gr}_{m,n} W$. It remains to notice that the previous (global) construction of the isotopy $\widehat{f}_t$ also works in the extension form. □

◀ **Example.** Suppose that $n < k < q$. Theorem 4.6.1 implies that any closed n-dimensional submanifold $V \subset \mathbb{R}^q$ whose tangent planes can be rotated into planes projecting non-degenerately on $\mathbb{R}^k \times 0 \subset \mathbb{R}^q$ along $0 \times \mathbb{R}^{q-k}$ can be perturbed via an isotopy so that its projection to $\mathbb{R}^k$ becomes an immersion. ▶

Two subbundles ξ, η of a bundle ζ are called *transversal* if the composition map $\xi \hookrightarrow \zeta \to \zeta/\eta$ is *surjective* if $\operatorname{rank}\xi + \operatorname{rank}\eta \geq \operatorname{rank}\zeta$, and *injective* if $\operatorname{rank}\xi + \operatorname{rank}\eta \leq \operatorname{rank}\zeta$.

4.6.2. (Generalization: closed submanifolds transversal to distributions) *Let ξ be a plane field (distribution) on a q-dimensional manifold W,* $\operatorname{codim}\xi = k$. *Let $n < k$. Then for any closed n-dimensional submanifold $V \subset W$ whose tangent bundle TV is homotopic inside TW to a subbundle $\tau \subset TW$ transversal to ξ, one can perturb V via an isotopy to make it transversal to ξ.*

◀ **Remark.** The relative and the parametric versions of Theorem 4.6.2 are also true. ▶

4.7. Approximation of differential forms by closed forms

A. Formal primitive of a differential form

Let us recall that any differential p-form ω on a manifold V can be considered as a section of the fibration $\Lambda^p V \to V$. In particular, 1-forms are sections of the fibration $\Lambda^1 V = T^* V \to V$.

The *exact* p-forms on V and the holonomic sections of the fibration

$$(\Lambda^{p-1}V)^{(1)} \to V$$

are closely related to each other. Indeed, the exterior differentiation

$$\mathrm{Sec}\,\Lambda^{p-1}V \xrightarrow{d} \mathrm{Sec}\,\Lambda^p V$$

can be written as the composition

$$\mathrm{Sec}\,\Lambda^{p-1}V \xrightarrow{J^1} \mathrm{Sec}\,(\Lambda^{p-1}V)^{(1)} \xrightarrow{\widetilde{D}} \mathrm{Sec}\,\Lambda^p V$$

where the map $\widetilde{D}$ is induced by a homomorphism of bundles over V

$$(\Lambda^{p-1}V)^{(1)} \xrightarrow{D} \Lambda^p V$$

which is called the *symbol* of the operator d. For example, for $p = 2$ the fiber of the first (affine) bundle $(\Lambda^{p-1}V)^{(1)} \to V$ is equivalent to the space of $n \times n$ matrices, the fiber of the second (vector) bundle $\Lambda^p V \to V$ is equivalent to the space of skew-symmetric $n \times n$ matrices and $D(A) = A - A^T$.

The map $D : (\Lambda^{p-1}V)^{(1)} \to \Lambda^p V$ is an affine fibration. In particular, any section $\omega : V \to \Lambda^p V$ can be lifted up in a homotopically canonical way to a section $F_\omega : V \to (\Lambda^{p-1}V)^{(1)}$ such that $D \circ F_\omega = \omega$. It is useful to think of F_ω as a *formal primitive* of ω. Therefore we can say that *any p-form has a formal primitive* or that any p-form is *formally exact.*

Note that there are no restrictions on the underlying section $\mathrm{bs}\,F_\omega$. In other words, given an arbitrary $(p-1)$-form $\alpha : V \to \Lambda^{p-1}V$ one can construct a formal primitive F_ω such that $\mathrm{bs}\,F_\omega = \alpha$.

B. Approximation of differential forms by closed forms

The theorems which we formulate below are rather technical. An important application will be given further in 10.2.

4.7.1. (Approximation by exact forms) *Let $K \subset V$ be a polyhedron of codimension ≥ 1 and ω a p-form. Then there exists an arbitrarily C^0-small diffeotopy $h^\tau : V \to V$ such that ω can be C^0-approximated near $\widetilde{K} = h^1(K)$ by an exact p-form $\widetilde{\omega} = d\widetilde{\alpha}$. Moreover, given a $(p-1)$-form α on V, one can chose $\widetilde{\alpha}$ to be C^0-close to α near $\widetilde{K}$.*

Proof. Take a formal primitive F_ω for ω such that $\mathrm{bs}\,F_\omega = \alpha$, choose its holonomic approximation $J^1_{\widetilde{\alpha}}$ along $\widetilde{K} = h^1(K) \subset V$, where h^τ is an (arbitrarily) C^0-small diffeotopy, and extend $\widetilde{\alpha}$ to the whole manifold V. Then $\widetilde{\omega} = d\widetilde{\alpha}$ is the desired exact form. □

Proposition 4.7.1 implies

4.7.2. (Approximation by closed forms) *Let $K \subset V$ be a polyhedron of codimension ≥ 1. Let ω be a p-form on V and $a \in H^p(V)$ a fixed cohomology class. Then there exists an arbitrarily C^0-small diffeotopy*

$$h^\tau : V \to V,\ t \in [0,1]$$

such that ω can be C^0-approximated near $\widetilde{K} = h^1(K)$ by a closed p-form $\widetilde{\omega} \in a$.

Indeed, one can take a closed form $\Omega \in a$, apply the previous proposition to the form $\theta = \omega - \Omega$ and then take $\widetilde{\omega} = \widetilde{\theta} + \Omega$.

The parametric versions of 4.7.1 and 4.7.2 are also valid. In particular,

4.7.3. (Parametric approximation by exact forms) *Let $K \subset V$ be a polyhedron of codimension ≥ 1 and $\{\omega_u\}_{u \in D^k}$ a family of p-forms such that $\{\omega_u = d\alpha_u\}_{u \in \partial D^k}$. Then there exists a family of arbitrarily C^0-small diffeotopies*

$$\{h_u^\tau : V \to V,\ t \in [0,1]\}_{u \in D^k}\,, \quad \textit{where}\ \{h_u^\tau = \mathrm{Id}_V,\ t \in [0,1]\}_{u \in \partial D^k}$$

such that $\{\omega_u\}_{u \in D^k}$ can be C^0-approximated near $\widetilde{K}_u = h_u^1(K)$ by a family of exact p-forms $\{\widetilde{\omega}_u = d\widetilde{\alpha}_u\}_{u \in D^k}$ such that $\{\widetilde{\alpha}_u = \alpha_u\}_{u \in \partial D^k}$. Moreover, given a family of $(p-1)$-forms $\{\alpha_u\}_{u \in D^k}$ on V, which extends the family $\{\alpha_u\}_{u \in \partial D^k}$, one can chose the family $\{\widetilde{\alpha}_u\}_{u \in D^k}$ to be C^0-close to $\{\alpha_u\}_{u \in D^k}$ near $\widetilde{K}_u$.

The proof is analogous to the proof of 4.7.1

4.7.4. (Parametric approximation by closed forms) *Let $a \in H^p(V)$ be a fixed cohomology class. Let $K \subset V$ be a polyhedron of codimension ≥ 1 and $\{\omega_u\}_{u \in D^k}$ a family of p-forms such that $\{d\omega_u = 0\,,\ \omega_u \in a\}_{u \in \partial D^k}$. Then there exists a family of arbitrarily C^0-small diffeotopies*

$$\{h_u^\tau : V \to V,\ t \in [0,1]\}_{u \in D^k}\,, \quad \textit{where}\ \{h_u^\tau = \mathrm{Id}_V,\ t \in [0,1]\}_{u \in \partial D^k}$$

such that $\{\omega_u\}_{u \in D^k}$ can be C^0-approximated near $\widetilde{K}_u = h_u^1(K)$ by a family of closed p-forms $\{\widetilde{\omega}_u \in a\}_{u \in D^k}$ such that $\{\widetilde{\omega}_u = \omega_u\}_{u \in \partial D^k}$.

Proof. The convexity of the space of closed p-forms on V which represent the class $a \in H^p(V)$ allows us to construct a family of closed forms $\{\Omega_u \in a\}_{u \in D^k}$ which extends the family $\{\omega_u\}_{u \in \partial D^k}$. Hence, it remains to apply 4.7.3 to $\{\theta_u = \omega_u - \Omega_u\}_{u \in D^k}$ and then take $\{\widetilde{\omega}_u = \widetilde{\theta}_u + \Omega_u\}_{u \in D^k}$. □

Part 2

Differential Relations and Gromov's h-Principle

Chapter 5

Differential Relations

5.1. What is a differential relation?

The language of jets is a vehicle for extending the Cartesian geometrization of algebraic equations to differential equations.

A *differential relation*, or *condition* of order r imposed on sections $f : V \to X$ of a fibration $X \to V$ is a subset $\mathcal{R}$ of the jet space $X^{(r)}$.

◀ **Example: Partial differential equations**

Any system of ordinary ($n = 1$), or partial ($n > 1$) differential equations

$$\Psi(x, f, D^\alpha f) = 0$$

imposed on unknown functions

$$y_j = f_j(x_1, \ldots, x_n),\ j = 1, \ldots, q,$$

and their derivatives

$$D^\alpha f_j = \frac{\partial^{|\alpha|} f_j}{\partial x_1^{\alpha_1}, \ldots, \partial x_n^{\alpha_n}},\ \ \alpha = (\alpha_1, \ldots, \alpha_n),\ |\alpha| = \alpha_1 + \cdots + \alpha_n \le r,$$

may be considered as a differential relation $\mathcal{R}$ in the r-jet space $J^r(\mathbb{R}^n, \mathbb{R}^q)$, defined by the system of "algebraic" (=non-differential) equations

$$\Psi(x, y, z_\alpha) = 0,$$

where the variables

$$x = (x_1, \ldots, x_n),\ y = (y_1, \ldots, y_q), \quad \text{and}\ \ z_\alpha = (z_{1,\alpha}, \ldots, z_{q,\alpha})$$

are coordinates in $J^r(\mathbb{R}^n, \mathbb{R}^q)$. This way any system of differential equations can be thought of as a subset of the jet space $J^r(\mathbb{R}^n, \mathbb{R}^q)$. ▶

Roughly speaking, differential equations and system of differential equations correspond to submanifolds of codimension ≥ 1 in the jet-space $J^1(\mathbb{R}^n, \mathbb{R}^q)$, while (strict) differential inequalities correspond to open subsets.

◀ **Exercise.** Draw differential relations in $J^1(\mathbb{R}, \mathbb{R})$, which correspond to the differential equation $y' = y^2$ and the differential inequality $y' \geq y^2$. ▶

◀ **Example: Immersions and submersions**

Let V and W be smooth manifolds, $n = \dim V$ and $q = \dim W$.

Let $n \leq q$. Let us recall that a map $f : V \to W$ is called an *immersion* if $\operatorname{rank} f = n$ everywhere on V, i.e. the differential $d_f : TV \to TW$ is fiberwise injective. The implicit function theorem implies that any immersion is locally equivalent to the inclusion $\mathbb{R}^n \hookrightarrow \mathbb{R}^q$.

The *immersion relation* $\mathcal{R}_{\text{imm}} \subset J^1(V, W)$ over each point $x = (v, w) \in V \times W$ consists of monomorphisms $T_v V \to T_w W$ or, equally, of (non-vertical) n-planes $P_x \subset T(V \times W)$ such that

$$\dim (P_x \cap T(v \times W)) = 0 \,.$$

Locally with respect to a trivialization

$$\varphi : \{U_1 \times U_2 \subset V \times W\} \to \mathbb{R}^n \times \mathbb{R}^n$$

the relation $\mathcal{R}_{\text{imm}}$ over each point $x = (v, w) \in V \times W$ consists of matrices $A \in M_{q \times n}$ of rank $= n$.

Let $n \geq q$. A map $f : V \to W$ is called a *submersion* if $\operatorname{rank} f = q$ everywhere on V, i.e. the differential $d_f : TV \to TW$ is fiberwise surjective. The implicit function theorem implies that any submersion is locally equivalent to the projection $\mathbb{R}^n \to \mathbb{R}^q$.

The *submersion relation* $\mathcal{R}_{\text{sub}} \subset J^1(V, W)$ over each point $x = (v, w) \in V \times W$ consists of epimorphisms (fiberwise surjective bundle homomorphisms) $T_v V \to T_w W$ or, equally, (non-vertical) n-planes $P_x \subset T(V \times W)$ such that

$$\dim (P_x \cap T(v \times W)) = n - q \,.$$

Locally, with respect to a trivialization

$$\varphi : \{U_1 \times U_2 \subset V \times W\} \to \mathbb{R}^n \times \mathbb{R}^n \,,$$

the relation $\mathcal{R}_{\text{sub}}$ over each point $x = (v, w) \in V \times W$ consists of matrices $A \in M_{q \times n}$ of rank $= q$. ▶

Note that for $n = q$

immersion = submersion = locally diffeomorphic map.

5.2. Open and closed differential relations

A differential relation $\mathcal{R} \subset X^{(r)}$ is called *open* or *closed* if it is open or closed as a subset of the jet-space $X^{(r)}$.

As we already mentioned above, closed subsets $\mathcal{R}$ which are submanifolds (or, more generally, stratified subsets) of positive codimension correspond to systems of differential equations. Such a relation is called *determined, overdetermined*, or *underdetermined* depending on whether $\operatorname{codim} \mathcal{R} = q$, $> q$, or $< q$. This classification corresponds to the usual classification of systems of (differential or algebraic) equations.

◀ **Examples.** The Laplace equation $\Delta f = 0$ defines a closed determined differential relation in $J^2(\mathbb{R}^n, \mathbb{R}^1)$. The differential equation $\sum \dot{x}_i^2 = 1$ defines a closed underdetermined differential relation in $J^1(\mathbb{R}, \mathbb{R}^n)$. The relations $\mathcal{R}_{\text{imm}}$ and $\mathcal{R}_{\text{sub}}$ are open differential relations in $J^1(V, W)$. ▶

◀ **Exercises**

1. Let $X = \Lambda^1\mathbb{R}^n$. The differential relation $\mathcal{R}_{\text{clo}} \subset (\Lambda^1\mathbb{R}^n)^{(1)}$ defines closed 1-forms on $\mathbb{R}^n$. When this relation is underdetermined? determined? overdetermined?

2. The differential relation $\mathcal{R}_{\text{iso}} \subset J^1(\mathbb{R}^n, \mathbb{R}^q)$ defines *isometric* immersions $f : \mathbb{R}^n \to \mathbb{R}^q$, i.e. $f^*h = g$ where g and h are standard metrics on $\mathbb{R}^n$ and $\mathbb{R}^q$. When this relation is underdetermined? determined? overdetermined? ▶

An open differential relation arises, for example, when one tries to find ε-approximate solutions of a closed differential relation $\mathcal{R}$. In this case our open relation $\mathcal{R}_\varepsilon$ is the ε-neighborhood of $\mathcal{R} \subset X^{(r)}$.

Another rich source of open differential relations is supplied by singularity theory, where one tries to construct functions, maps or sections for which certain expressions involving derivatives never vanish, and thus we are led to a differential relation $\mathcal{R}$ which is the complement $X^{(r)} \setminus \Sigma$ of a submanifold (or, more generally, of a stratified subset) $\Sigma \subset X^{(r)}$. This Σ is usually called *a singularity*. Solving $\mathcal{R} = X^{(r)} \setminus \Sigma$ means finding Σ-*non-singular* holonomic sections $V \to X^{(r)}$.

Suppose, for example, that we are interested in finding immersions (submersions) $V \to W$. The differential relation $\mathcal{R}_{\text{imm}}$ ($\mathcal{R}_{\text{sub}}$) is the complement of the stratified subset $\Sigma^1 \subset J^1(V, W)$.

5.3. Formal and genuine solutions of a differential relation

Any section $F : V \to \mathcal{R} \subset X^{(r)}$ is called a *formal solution* of the differential relation $\mathcal{R}$.

◀ **Examples**

1. A formal solution to a system of differential equations is a solution of the underlying system of "algebraic" equations obtained by substituting derivatives by new independent functions.

2. A formal solution of the immersion relation $\mathcal{R}_{\mathrm{imm}}$ is a monomorphism (=fiberwise injective bundle homomorphism) $TV \to TW$. A formal solution of the submersion relation $\mathcal{R}_{\mathrm{sub}}$ is an epimorphism (=fiberwise surjective bundle homomorphism) $TV \to TW$. ▶

A (genuine) *solution* of a differential relation $\mathcal{R} \subset X^{(r)}$ is a section $f : V \to X$ such that $J^r_f(V) \subset \mathcal{R}$. Alternatively, we can define solutions of $\mathcal{R}$ as *holonomic sections* $F = J^r_f : V \to \mathcal{R}$. We will call the holonomic sections $V \to \mathcal{R} \subset X^{(r)}$ *r-extended solutions* or just *r-solutions*, when the distinction between the solutions of $\mathcal{R}$ as sections of X or X^r is not clear from the context.

We will denote the space of solutions of $\mathcal{R}$ by $\operatorname{Sol}\mathcal{R}$, the space of r-solutions of $\mathcal{R}$ by $\operatorname{Hol}\mathcal{R}$ and the space of formal solutions of $\mathcal{R}$ by $\operatorname{Sec}\mathcal{R}$. The r-jet extension gives a one-to-one correspondence $J^r : \operatorname{Sol}\mathcal{R} \to \operatorname{Hol}\mathcal{R}$.

◀ **Exercises**

1. Write down the formal and the genuine solutions of the differential equations $y' = y$ and $y' = y^2$. Draw these formal and genuine solutions in the jet space $J^1(\mathbb{R}, \mathbb{R})$.

2. Write down all formal solutions of the Laplace equation $\frac{\partial^2 f}{\partial x_1^2} + \frac{\partial^2 f}{\partial x_2^2} = 0$.

3. Write down the formal solutions of the isometry relation $\mathcal{R}_{\mathrm{iso}} \subset J^1(\mathbb{R}^2, \mathbb{R}^3)$. ▶

5.4. Extension problem

Consider the following "boundary value problem". Let $\mathcal{R} \subset X^{(r)}$ be a differential relation and f its solution over $\mathcal{O}p\, B \subset V$ (i.e. over an open neighborhood of B). Given a bigger subset $A \supset B$ we want to extend the solution f as a solution of $\mathcal{R}$ over $\mathcal{O}p\, A$. A *formal solution* of the extension problem is a section F over $\mathcal{O}p\, A$ which coincides with J^r_f over $\mathcal{O}p\, B$. A special case of the extension problem, when $A \simeq D^k$ and $B = \partial A \simeq S^{k-1}$ is especially important for us.

In the sequel we use the term *solution* (or *formal solution*) of $\mathcal{R}$ also in the sense of a *solution (formal solution)* of an extension problem for $\mathcal{R}$. Global solutions of $\mathcal{R}$ (over V) correspond to the case $A = V,\ B = \varnothing$.

5.5. Approximate solutions to systems of differential equations

Let $X \to V$ be a fibration and $\mathcal{R} \subset X^{(r)}$ a closed differential relation of positive codimension. A section $f : V \to X$ is called an *ε-approximate solution* of $\mathcal{R}$ if f is a solution of the open relation $\mathcal{R}_\varepsilon = U_\varepsilon(\mathcal{R})$, where $U_\varepsilon(\mathcal{R})$ is an ε-neighborhood of $\mathcal{R}$ in $X^{(r)}$. The Holonomic Approximation Theorem 3.1.1 can be reformulated in the following way:

5.5.1. *Let $A \subset V$ be a polyhedron of positive codimension and $F : \mathcal{O}p\, A \to \mathcal{R}$ a formal solution of $\mathcal{R}$. Then for arbitrarily small $\delta, \varepsilon > 0$ there exist a δ-small diffeotopy $h^\tau : V \to V$, $\tau \in [0,1]$, and an ε-approximate solution $\widetilde{f}$ of $\mathcal{R}$ over $\mathcal{O}p\, h^1(A)$ such that*

$$||J^r_{\widetilde{f}} - F|_{\mathcal{O}p\, h^1(A)}||_{C^0} < \varepsilon\,.$$

Figure 5.1. Hypersurfaces S, $\widetilde{S}$ and their neighborhoods $\mathcal{O}p\ S$ and $\mathcal{O}p\ \widetilde{S}$.

◀ **Example.** Let $\mathcal{R}$ be a system of partial differential equations in $\mathbb{R}^n$. Suppose we are given initial data D along a hypersurface $S \in \mathbb{R}^n$ and we want to find an approximate solution of the *local* Cauchy problem for $\mathcal{R}$ near S. Suppose that there exists an extension $\bar{D}$ of D on $\mathcal{O}p\,S$ such that $\bar{D}$ is a *formal* solution of $\mathcal{R}$. Then Theorem 5.5.1 implies that for any $\varepsilon > 0$ there exists an ε-approximate solution of $\mathcal{R}$ near a slightly perturbed hypersurface $\widetilde{S} = h(S)$. In other words, instead of an approximate solution near the desired S we can find an approximate solution near a C^0-small perturbed (in the direction normal to S) hypersurface $\widetilde{S}$ (see Fig.5.1). ▸

Chapter 6

Homotopy Principle

6.1. Philosophy of the h-principle

Existence of a formal solution is a *necessary* condition for the solvability of a differential relation $\mathcal{R}$, and thus before trying to solve $\mathcal{R}$ one should check whether $\mathcal{R}$ admits a formal solution. The problem of finding formal solutions is of purely homotopy-theoretical nature. This problem may be simple or highly non-trivial, but in any case it is important to treat the homotopical problem first, and look for genuine solutions only after existence of formal solutions has been established.

◀ **Exercises**

1. Let $\mathcal{R} \subset J^1(\mathbb{R}, \mathbb{R})$ be defined by a non-equality $f' \neq 0$. Let $A = [0, 2\pi]$, $B = \partial A$, and $f|_{\mathcal{O}p\, B} = \sin x$. Find a formal solution of the extension problem and prove that there are no genuine solutions.

2. Let $\mathcal{R} \subset J^1(\mathbb{R}^2, \mathbb{R})$ be defined by a non-equality $\operatorname{grad} f \neq 0$. Let $f|_{\mathcal{O}p\, \partial D^2} = x_1^2 - x_2^2$. Prove that the extension problem does not even have formal solutions.

3. Let $\mathcal{R}$ be the same relation as in the previous Exercise. Let A be the annulus $a \leq r \leq b$ where $r = \sqrt{x_1^2 + x_2^2}$ and $B = \partial A$. Let $f|_{\mathcal{O}p\, B} = (r - r_0)^2$ where $r_0 = (a + b)/2$. Find a formal solution of the extension problem and prove that there are no genuine solutions.

4. Consider the relation $\mathcal{R}_{\mathrm{imm}} \subset J^1(S^1, \mathbb{R}^2)$. When are two immersions $S^1 \to \mathbb{R}^2$ homotopic in $\operatorname{Sec} \mathcal{R}_{\mathrm{imm}}$? ▶

It seems at first thought that existence of a formal solution cannot be *sufficient* for the genuine solvability of $\mathcal{R}$. As we already said above, finding a formal solution is an algebraic, or homotopy-theoretical problem, which is a dramatic simplification of the original differential problem. Thus it came as a big surprise when it was discovered in the second half of the XX century that here exist large and geometrically interesting classes of differential relations for which the solvability of the formal problem turned out to be sufficient for genuine solvability. Moreover, for many of these relations the spaces of formal and genuine solutions turned out to be much closer related then one could expect. This property was formalized in [**GE71**] and [**Gr71**] as the following

- **Homotopy principle (h-principle).** *We say that a differential relation $\mathcal{R}$ satisfies the* h**-principle**, *or that the h-principle holds for solutions of $\mathcal{R}$, if every formal solution of $\mathcal{R}$ is homotopic in* $\operatorname{Sec}\mathcal{R}$ *to a genuine solution of $\mathcal{R}$.*

A similar definition can be given for solutions of an extension problem for $\mathcal{R}$ and for families of solutions. For example, we say that

- *$\mathcal{R}$ satisfies the* **one-parametric** h**-principle** *if every family of formal solutions $\{f_t\}_{t\in I}$ of $\mathcal{R}$ which joins two genuine solutions f_0 and f_1 can be deformed inside* $\operatorname{Sec}\mathcal{R}$, *keeping f_0 and f_1 fixed, into a family $\{\widetilde{f_t}\}_{t\in I}$ of genuine solutions of $\mathcal{R}$.*

In fact, it is useful to consider different "degrees" of the h-principle when one want to establish closer and closer connections between formal and genuine solutions. For instance, different forms of the h-principle may include some approximation and extension properties. We will discuss the different flavors of the h-principle in the next chapter.

◀ **Exercise (H. Whitney, [Wh37]).** Prove the one-parametric h-principle for $\mathcal{R}_{\mathrm{imm}} \subset J^1(S^1, \mathbb{R}^2)$. ▶

Hints.

(a) Let $f_0, f_1 : S^1 \to \mathbb{R}^2$ be two immersions and (f_τ, v_τ) a homotopy of formal immersions which connects $(f_0, \dot{f}_0)$ and $(f_1, \dot{f}_1)$, i.e. $v_\tau(t) \neq 0$. Consider some extensions $\bar{f}_0, \bar{f}_1 : S^1 \times (-\varepsilon, \varepsilon) \to \mathbb{R}^2$ of f_0, f_1 and an extension $\bar{F}_\tau : S^1 \times (-\varepsilon, \varepsilon) \to J^1(S^1 \times (-\varepsilon, \varepsilon), \mathbb{R}^2)$ of F_τ and apply the Parametric Holonomic Approximation Theorem 3.1.2 (compare 4.2).

(b) (Whitney's proof) We may assume from the very beginning that the lengths of the curves f_0 and f_1 are equal to 1 and $|v_\tau(t)| = 1$ for all $t \in [0,1]/\{0,1\} \simeq S^1$ and $\tau \in [0,1]$. Consider the family of immersions $g_\tau : I \to$

$\mathbb{R}^2$, $\tau \in I$, given by

$$g_\tau(t) = f_\tau(0) + \int_0^t v_\tau(\sigma) d\sigma \,, \quad t \in [0,1] \,.$$

Try to rearrange things by translation of v_τ in order to get closed regular curves g_τ for all τ.

◀ **Remark.** S. Smale in [**Sm59**] generalized Whitney's theorem to the case of immersions $S^n \to \mathbb{R}^q$. His "covering homotopy" method was different from Whitney's method. In fact, Whitney's method is closer, in some sense, to the idea of convex integration, see Part IV of the book. ▶

◀ **Exercise.** Prove the h-principle for the differential equation $y' = y$ and disprove it for the differential equation $y' = y^2$ (in both cases we consider global solutions). Disprove the h-principle in both cases for an extension problem with $A = D^1$ and $B = \partial D^1$. ▶

The examples in the last exercise are trivial and, of course, not typical for the situations where the h-principle becomes a useful notion. In fact, the h-principle is rather useless for the classical theory of (ordinary or partial) differential equations because there, as a rule, it fails or holds for some trivial, or at least well known reasons, as in the above examples.

By contrast, for many differential relations rooted in topology and geometry the notion of h-principle appears to be fundamental, *whether it holds or not*. For a given differential relation *a priori* we often have no obvious reasons both for the validity or failure of the h-principle. Paradoxically, it appeared that sometimes we need extremely sophisticated tools for *disproving* the h-principle. For example, modern Symplectic geometry was born in a long battle for establishing the borderline between the areas where the h-principle holds and where it fails. Since the beginning of the eighties the Symplectic Rigidity army scored a lot of victories which brought to life the whole new area of *Symplectic topology*. However, there were also several amazing unexpected breakthroughs on the Flexibility side (see Chapter 11 below). In fact, it is still possible that in spite of great recent successes of Symplectic topology, the world of Symplectic Rigidity is just a small island floating in the Flexible Symplectic Ocean.

This book is devoted to two general methods of *proving* the h-principle: *holonomic approximation* and *convex integration.*

6.2. Different flavors of the h-principle

Let $\mathcal{R} \subset X^{(r)}$ be a differential relation.

A. Parametric h-principle

We say that:

- *the (multi) parametric h-principle* holds for $\mathcal{R}$ if for every relative spheroid
$$\varphi_0 : (D^k, S^{k-1}) \to (\operatorname{Sec}\mathcal{R},\ \operatorname{Hol}\mathcal{R})\ ,\ k = 0, 1, \dots$$
there exists a fixed on S^{k-1}, homotopy
$$\varphi_t : (D^k, S^{k-1}) \to (\operatorname{Sec}\mathcal{R}\,, \operatorname{Hol}\mathcal{R}),\ \ t \in [0, 1],$$
such that $\varphi_1(D^k) \subset \operatorname{Hol}\mathcal{R}$. In other words, the inclusion $\operatorname{Hol}\mathcal{R} \to \operatorname{Sec}\mathcal{R}$ is a *weak homotopy equivalence.*

Using the language of homotopy groups we can say that the parametric h-principle for a differential relation $\mathcal{R}$ means that $\pi_k(\operatorname{Sec}\mathcal{R}\,, \operatorname{Hol}\mathcal{R}) = 0\,, k = 0, 1, \dots$. In particular, the map $\operatorname{Hol}\mathcal{R} \to \operatorname{Sec}\mathcal{R}$ induces an isomorphism $\pi_0(\operatorname{Hol}\mathcal{R}) \to \pi_0(\operatorname{Sec}\mathcal{R})$. The epimorphism on π_0 means that any formal solution is homotopic (in $\operatorname{Sec}\mathcal{R}$) to a genuine solution, and the monomorphism on π_0 means that two genuine solutions which are homotopic in $\operatorname{Sec}\mathcal{R}$ are also homotopic in $\operatorname{Hol}\mathcal{R}$.

◀ Remark: Homotopy equivalence vs. weak homotopy equivalence

An infinite-dimensional version of the J.H.C. Whitehead theorem (see [**Pa66**] and [**Ee66**]) implies that for metrizable Fréchet manifolds *weak homotopy equivalence implies the usual homotopy equivalence.* In particular, the spaces of sections $\operatorname{Sec}\mathcal{R}$ and $\operatorname{Hol} R$ are metrizable Fréchet manifolds and hence the parametric h-principle for $\mathcal{R}$ implies that the inclusion $\operatorname{Hol}\mathcal{R} \to \operatorname{Sec}\mathcal{R}$ is a homotopy equivalence. It allows us to skip the word "weak" in all our statements about the parametric h-principle. However, in all the applications one usually needs just *weak* homotopy equivalence. Hence, the reader who feels uncomfortable with this infinite-dimensional argumentation may just reinsert the word "weak" into all statements about homotopy equivalence. ▶

B. Local h-principle

Let A be an arbitrary subset of V. We say that:

- *the (local) h-principle holds for $\mathcal{R}$ near A*, if for every formal solution $F_0 : \mathcal{O}p\, A \to \mathcal{R}$ there exists a homotopy $F_t : \mathcal{O}p\, A \to \mathcal{R},\ t \in [0, 1]$, such that F_1 is a genuine solution;

- *the parametric (local) h-principle holds for $\mathcal{R}$ near A* if for every relative spheroid

  $$\varphi_0 : (D^k, S^{k-1}) \to (\mathrm{Sec}_{\mathcal{O}p\,A}\mathcal{R},\ \mathrm{Hol}_{\mathcal{O}p\,A}\mathcal{R})\ ,\ k = 0, 1, \dots ,$$

  there exists a fixed on S^{k-1} homotopy

  $$\varphi_t : (D^k, S^{k-1}) \to (\mathrm{Sec}_{\mathcal{O}p\,A}\mathcal{R}\,, \mathrm{Hol}_{\mathcal{O}p\,A}\mathcal{R}),\ t \in [0, 1],$$

  such that $\varphi_1(D^k) \subset \mathrm{Hol}_{\mathcal{O}p\,A}\mathcal{R}$.

C. Relative h-principle, or h-principle for extensions

For the subsets $B \subset A \subset V$ we denote by $\mathrm{Sec}_{\mathcal{O}p\,(A,B)}\mathcal{R}$ the space of formal solutions $F : \mathcal{O}p\,A \to \mathcal{R}$ which are holonomic near B. We say that:

- *the (relative) h-principle holds for $\mathcal{R}$ near the pair (A, B)* if for every formal solution $F_0 \in \mathrm{Sec}_{\mathcal{O}p\,(A,B)}\mathcal{R}$ there exists a homotopy through formal solutions

  $$F_t \in \mathrm{Sec}_{\mathcal{O}p\,(A,B)}\mathcal{R},\ t \in [0, 1],$$

  such that $F_t|_{\mathcal{O}p\,B} = F|_{\mathcal{O}p\,B}$ for all $t \in [0, 1]$ and F_1 is a genuine solution;

- *the parametric (relative) h-principle holds for $\mathcal{R}$ near the pair (A, B)*, if for every relative spheroid

  $$\varphi_0 : (D^k, S^{k-1}) \to (\mathrm{Sec}_{\mathcal{O}p\,(A,B)}\mathcal{R},\ \mathrm{Hol}_{\mathcal{O}p\,A}\mathcal{R})\ ,\ k = 0, 1, \dots ,$$

  there exists a fixed on S^{k-1} homotopy

  $$\varphi_t : (D^k, S^{k-1}) \to (\mathrm{Sec}_{\mathcal{O}p\,(A,B)}\mathcal{R}\,, \mathrm{Hol}_{\mathcal{O}p\,A}\mathcal{R}),\ t \in [0, 1],$$

  such that $\varphi_1(D^k) \subset \mathrm{Hol}_{\mathcal{O}p\,A}\mathcal{R}$ and for every $p \in D^k$ the homotopy $\varphi_t(p) : \mathcal{O}p\,A \to \mathcal{R}$ is fixed near B.

D. $\mathbf{C^0}$-dense h-principle

We say that

- the C^0-*dense h-principle* holds for $\mathcal{R}$ if the (usual) h-principle holds for $\mathcal{R}$ and if for every formal solution $F_0 : V \to \mathcal{R}$ and an arbitrarily small neighborhood $U \subset X$ of the underlying section $f_0 = \mathrm{bs}\,F_0$ the homotopy $F_t,\ t \in [0, 1]$, in $\mathcal{R}$ which brings F_0 to a genuine solution F_1 can be chosen in such a way that $\mathrm{bs}\,F_t(V) \subset U\ ,\ t \in [0, 1]$.

In a similar way one defines the C^0-dense versions of all previously defined h-principles.

E. Fibered differential relations

Sometimes when we are working with the parametric situation the differential relation itself may also depend on the parameter. Let us formulate the corresponding definition in this situation.

Let P be the space of parameters.

A map

$$f : P \times A \to P \times B; \quad (p, a) \mapsto (p, f_p(a))$$

is called *fibered* (over P) map.

Any subset $\mathcal{R} \subset P \times X^{(r)}$ is called a *fibered differential relation* imposed on the *fibered* (over P) sections

$$f : P \times V \to P \times X; \quad (p, v) \mapsto (p, f_p(v))$$

which are continuously depending on $p \in P$. *Formal solutions* of $\mathcal{R}$ are sections $P \times V \to \mathcal{R} \subset P \times X^{(r)}$ and (extended) *solutions* of $\mathcal{R}$ are fiberwise holonomic sections $P \times V \to \mathcal{R}$.

The parametric h-principle can be reformulated in the following *fibered* version. We say that

- a *fibered* h-principle holds for a fibered relation $\mathcal{R} \subset P \times X^{(r)}$ if every (fibered) formal solution $F_0 : P \times V \to P \times X^{(r)}$ is homotopic via homotopy of fibered formal solutions F_t to a fibered genuine solution $F_1 : P \times V \to \mathcal{R}$.

If P is a manifold with non-empty boundary ∂P then we usually assume that the formal solution F is holonomic over $\partial P \times V$ and require the homotopy F_t to be fixed near $\partial P \times V$. The parametric h-principle can now be reformulated as follows: the parametric h-principle holds for $\mathcal{R}$ if and only if for every $k = 0, 1, \dots$ the fibered h-principle holds for $\mathcal{R}_k = D^k \times \mathcal{R} \subset D^k \times X^{(r)}$.

Chapter 7

Open $\operatorname{Diff} V$-Invariant Differential Relations

7.1. $\operatorname{Diff} V$-invariant differential relations

Given a fibration $p : X \to V$ we will denote by $\operatorname{Diff}_V X$ the group of fiber-preserving diffeomorphisms $h_X : X \to X$, i.e. $h_X \in \operatorname{Diff}_V X$ if and only if there exists a diffeomorphism $h_V : V \to V$ such that the diagram

$$\begin{array}{ccc} X & \xrightarrow{h_X} & X \\ {\scriptstyle p}\big\downarrow & & \big\downarrow{\scriptstyle p} \\ V & \xrightarrow{h_V} & V \end{array}$$

commutes. Let $\pi : \operatorname{Diff}_V X \to \operatorname{Diff} V$ be the projection $h_X \mapsto h_V$. We are interested in situation when this arrow can be reversed, i.e. *when there exists a homomorphism* $j : \operatorname{Diff} V \to \operatorname{Diff}_V X$ *such that* $\pi \circ j = \mathrm{id}$. We call a fibration $X \to V$ together with a homomorphism j *natural* if such a lift exists. For instance, the trivial fibration $X = V \times W \to V$ is natural. Here $j(h_V) = h_V \times \mathrm{id}$. The tangent bundle $TV \to V$ is also natural. The corresponding lift here is provided by the differential $df : TV \to TV$ of a diffeomorphism $f : V \to V$. If a fibration $X \to V$ is natural then any fibration associated with it is natural as well. In particular, if $X \to V$ is natural then $X^{(r)} \to V$ is natural. The implied lift

$$j^r : \operatorname{Diff} V \to \operatorname{Diff}_V X^{(r)}\,, \quad h \mapsto j^r(h) = h_*$$

is defined here by the formula

$$h_*(s) = J^r_{j(h)\circ \bar{s}}(h(v))$$

where $s \in X^{(r)}$, $v = p^r(s) \in V$, and $\bar{s}$ is a local section near v which represents the r-jet s.

Note that $(h^{-1})_* = (h_*)^{-1}$. Set $h^* = h_*^{-1}$.

◄ **Example.** Let $X = \Lambda^p V$ be the exterior power of the cotangent bundle to V. Then any diffeomorphism $h : V \to V$ can be lifted on $\Lambda^p V$ as the exterior power $d^p h$ of the differential dh:

$$j(h) = d^p h : (v, \omega) \mapsto (h(p), \omega_h) \ , p \in V \ , \omega \in \Lambda_v^p V \ , \omega_h \in \Lambda_{h(v)}^p V \ ,$$

where the value of the form ω_h on the vectors $a_1, \dots, a_p \in T_{h(v)}$ is defined by the formula

$$\omega_h(a_1, \dots, a_p) = \omega(dh^{-1}(a_1), \dots, dh^{-1}(a_p)) \ \blacktriangleright$$

Given a natural fibration $X \to V$ a differential relation $\mathcal{R} \subset X^{(r)}$ is called Diff V-*invariant* if the action $s \mapsto h_* s$, $h \in \operatorname{Diff} V$, leaves $\mathcal{R}$ invariant. In other words, a differential relation $\mathcal{R}$ is Diff V-invariant if it can be defined in V-coordinate free form. Note that although the definition of a Diff V-invariant relation depends on the choice of the homomorphism j, this choice is fairly obvious in most interesting examples and we will not specify it.

The action $s \mapsto h_* s$ preserves the set of holonomic sections:

$$h_*(J_f^r) = J^r\left(j(h) \circ f \circ h^{-1}\right),$$

$f \in \operatorname{Sec} X$, $h \in \operatorname{Diff} V$. In particular, the group Diff V acts on the space $\operatorname{Sol} \mathcal{R}$ of solutions of an invariant differential relation $\mathcal{R}$.

◄ **Examples.** The relations $\mathcal{R}_{\mathrm{imm}}$ and $\mathcal{R}_{\mathrm{subm}}$ are Diff V-invariant. For any $A \subset \operatorname{Gr}_n W$ the relation $\mathcal{R}_A$ which defines A-directed maps $V \to W$ (see 4.5) is Diff V-invariant. ►

7.2. Local h-principle for open Diff V-invariant relations

7.2.1. (Local h-principle) *Let $X \to V$ be a natural fiber bundle. Then any open* Diff V*-invariant differential relation $\mathcal{R} \subset X^{(r)}$ satisfies all forms of the local h-principle near any polyhedron $A \subset V$ of positive codimension.*

Proof. First we will consider the non-parametric case, i.e. we will prove that

$$\pi_0(\operatorname{Sec}_{\mathcal{O}p\,A}\mathcal{R}, \operatorname{Hol}_{\mathcal{O}p\,A}\mathcal{R}) = 0\,.$$

We need to show that, given a section $F \in \operatorname{Sec}_{\mathcal{O}p\,A}\mathcal{R}$, there exists a section $G \in \operatorname{Hol}_{\mathcal{O}p\,A}\mathcal{R}$ which is homotopic in $\operatorname{Sec}_{\mathcal{O}p\,A}\mathcal{R}$ to F. According to Theorem 3.1.1 there exist an arbitrarily C^0-small diffeotopy $h^\tau : V \to V$,

$\tau \in [0,1]$, and a section $\widetilde{F} \in \mathrm{Hol}_{\mathcal{O}p\,h(A)}\mathcal{R}$ such that $\widetilde{F}$ is C^0-close to F over $\mathcal{O}p\,h(A)$. In particular, we may assume that the linear homotopy $\widetilde{F}^t$ between $F|_{\mathcal{O}p\,h^1(A)} = \widetilde{F}^0$ and $\widetilde{F} = \widetilde{F}^1$ lies in $\mathcal{R}$. The desired section $G \in \mathrm{Hol}_{\mathcal{O}p\,A}\mathcal{R}$ can then be defined by the formula $G = (h^1)^*\widetilde{F}$ where $(h^1)^* = (h^1)_*^{-1} : X^{(r)} \to X^{(r)}$ is the induced action of the "straightening" diffeomorphism $(h^1)^{-1}$ on the natural fibration $X^{(r)} \to V$. The required homotopy in $\mathcal{R}$, which connects F and G over a neighborhood of A, consist of two stages: first $(h^\tau)^*F$, $\tau \in [0,1]$, and then $(h^1)^*(\widetilde{F}^t)$, $t \in [0,1]$.

Note that although the holonomic approximation gives us a section $\widetilde{F}$ over $\mathcal{O}p\,h(A)$ which is C^0-close in $X^{(r)}$ to the initial formal solution F over $\mathcal{O}p\,h(A)$, it does not imply the same property for G over $\mathcal{O}p\,A$. Indeed, the C^0-closeness to F deadly fails for the straightened solution $(h^1)^*\widetilde{F}$ on $\mathcal{O}p\,A$. However, the C^0-approximation over $\mathcal{O}p\,A$ *in* X survives after straightening and this imply the *local* C^0*-dense* h-principle for $\mathcal{R}$ near A.

For the general parametric, relative and relative parametric cases we need to use the corresponding parametric/relative versions of the Holonomic Approximation Theorem 3.1.1. However, the proof for these cases differs only in notation. Let us consider, for example, the parametric case.

In order to prove the equality $\pi_m(\mathrm{Sec}_{\mathcal{O}p\,A}\mathcal{R}, \mathrm{Hol}_{\mathcal{O}p\,A}\mathcal{R}) = 0$, $m \geq 1$, we need to show that, given a family of sections $F_z \in \mathrm{Sec}_{\mathcal{O}p\,A}\mathcal{R}$, $z \in I^m$, $m = 0, \dots$ such that for $z \in \mathcal{O}p\,\partial I^m$ the section F_z is holonomic, there exists a family $G_z \in \mathrm{Hol}_{\mathcal{O}p\,A}\mathcal{R}$ which is homotopic in $\mathrm{Sec}_{\mathcal{O}p\,A}\mathcal{R}$ to the family F_z, $z \in I^m$, relative to ∂I^m. According to Theorem 3.1.1 there exist a family of arbitrarily C^0-small diffeotopies $h_z^\tau : V \to V$, $\tau \in [0,1]$, $z \in I^m$, and a family of sections $\widetilde{F}_z \in \mathrm{Hol}_{\mathcal{O}p\,h(A)}\mathcal{R}$ such that $h_z = \mathrm{Id}$ for $z \in \partial I^m$ and $\widetilde{F}_z$ is C^0-close to F over $\mathcal{O}p\,h_z^1(A)$. In particular, we may assume that the linear homotopy $\widetilde{F}_z^t$ between $F_z|_{\mathcal{O}p\,h_z^1(A)}$ and $\widetilde{F}_z$ lies in $\mathcal{R}$. The desired family of sections $G_z \in \mathrm{Hol}_{\mathcal{O}p\,A}\mathcal{R}$ can be then defined by the formula $G_z = (h_z^1)^*\widetilde{F}_z$ where $(h_z^1)^* = (h_z^1)_*^{-1} : X^{(r)} \to X^{(r)}$ is the induced action of the straightening diffeomorphisms $(h_z^1)^{-1}$ on the natural fibration $X^{(r)} \to V$. The required homotopy in $\mathcal{R}$ which connects F_z and G_z over a neighborhood of A consists of two stages: first $(h_z^\tau)^*F$, $\tau \in [0,1]$, and then $(h_z^1)^*(\widetilde{F}^t)$, $t \in [0,1]$. □

7.2.2. (Local h-principle implies global for open manifolds) *Let V be an open manifold and $X \to V$ a natural fibration. Let $\mathcal{R} \subset X^{(r)}$ be a* Diff V*-invariant differential relation. Then the parametric local h-principle for $\mathcal{R}$ implies the parametric global h-principle for $\mathcal{R}$.*

Proof. We consider only the non-parametric case, i.e. prove that

$$\pi_0(\mathrm{Sec}\,\mathcal{R}, \mathrm{Hol}\,\mathcal{R}) = 0\,.$$

The general parametric case differs only in notation. We need to show that, given a section $F \in \mathrm{Sec}\,\mathcal{R}$, there exists a section $G \in \mathrm{Hol}\,\mathcal{R}$ which is homotopic in $\mathrm{Sec}\,\mathcal{R}$ to F. Let $K \subset V$ be a core of the manifold V. The local h-principle near K implies the existence of a section $G_K \in \mathrm{Hol}_{\mathcal{O}p\,K}\mathcal{R}$ which is homotopic in $\mathrm{Sec}_{\mathcal{O}p\,K}\mathcal{R}$ to $F|_{\mathcal{O}p\,K}$. Let $g^\tau : V \to V$, $\tau \in [0,1]$, be an isotopy compressing V into a neighborhood U of K such that G_K is defined over U. The desired section $G \in \mathrm{Hol}\,\mathcal{R}$ can now be defined by the formula $G = (g^1)^* G_K$ where $(g^1)^* = (g^1)_*^{-1} : X^{(r)} \to X^{(r)}$ is the induced action of the "decompressing" diffeomorphism $(g^1)^{-1}$ on the natural fibration $X^{(r)} \to V$. The required homotopy in $\mathcal{R}$ which connects F and G consists of two stages: first $(g^\tau)^*F$, $\tau \in [0,1]$, and then $(g^1)^*G^t_K$, $t \in [0,1]$, where G^t_K is the homotopy which connects $F_{\mathcal{O}p\,K}$ and G_K in $\mathcal{R}$. □

◀ **Remark.** Note that the local C^0-*dense* h-principle near K does not survive after an expansion and hence we cannot derive the global C^0-dense h-principle for $\mathcal{R}$ from the local C^0-dense h-principle. ▶

Theorems 7.2.1 and 7.2.2 imply

7.2.3. (Gromov [Gr69]) *Let V be an open manifold and $X \to V$ a natural fiber bundle. Then any open* Diff$V$*-invariant differential relation $\mathcal{R} \subset X^{(r)}$ satisfies the parametric h-principle. In particular immersions, submersions, k-mersions (maps of rank $\geq k$), and immersions $V \to W$ directed by an open set $A \subset \mathrm{Gr}_n(W)$ (see 4.5) satisfy the parametric h-principle as long as the underlying manifold V is open.*

The *relative* h-principle in this situation holds in the following version:

7.2.4. *Let $\mathcal{R} \subset X^{(r)}$ be an open* Diff V*-differential relation over an open manifold V. Let $B \subset V$ be a closed subset such that each connected component of the complement $V \setminus B$ has an exit to ∞. Then the relative parametric h-principle holds for $\mathcal{R}$ and the pair (V, B).*

Chapter 8

Applications to Closed Manifolds

8.1. Microextension trick

The *microextension trick*, which goes back to M. Hirsch, allows sometimes to reformulate problems about closed manifolds in terms of open manifolds. For example if $\dim W > \dim V$ then a construction of an immersion $V \to W$ homotopic to a map $f : V \to W$ is equivalent to a construction of an immersion $E \to W$ where E is the total space of the normal bundle to TV in f^*TW. The manifold E is open and hence the h-principle 7.2.3 applies. Above we already used the microextension trick for directed embeddings of closed manifolds, see Theorem 4.6.1.

8.2. Smale-Hirsch h-principle

The h-principle for immersions $V \to W$ obviously fails if V is closed and $n = q$: a closed n-dimensional manifold never admits an immersion into $\mathbb{R}^n$ even if it is parallelizable, which is equivalent to the existence of a formal solution of the immersion problem.

◀ **Exercise.** Prove that an immersion of a neighborhood of ∂D^2 into $\mathbb{R}^2$ which is shown on Fig.8.1 cannot be extended to an immersion of the disk D^2 into $\mathbb{R}^2$, while it extends to a formal immersion. ▶

However, in the case $n < q$ one gets the C^0-dense parametric h-principle via the microextension trick.

8.2.1. (Hirsch, [Hi59]) *The parametric C^0-dense h-principle holds for immersions of an n-dimensional manifold V into a manifold W of dimension $q > n$.*

Figure 8.1. An immersion $S^1 \times (-\varepsilon, \varepsilon) \to \mathbb{R}^2$ which can not be extended to an immersion $D^2 \to \mathbb{R}^2$ although it has a formal extension to D^2

Proof. In the non-parametric case let F be a formal solution of the differential relation $\mathcal{R}_{imm} \subset J^1(V, W)$. Set $f = \operatorname{bs} F$. The homomorphism F identifies TV with a rank $= n$ subbundle λ of the induced bundle f^*TW. Let ν be the normal bundle to λ in TW with respect to a choice of a Riemannian metric on W, N be the total space of the bundle ν and $\pi : N \to V$ the projection. Then $TN = \pi^*TV \oplus \pi^*\nu$, and hence F canonically lifts to a fiberwise isomorphism $\widetilde{F} : TN \to TW$. The restriction of an immersion $N \to W$ to the zero-section $V \subset N$ is an immersion $V \to W$, and hence applying C^0-dense local h-principle near V we get the required h-principle for immersions $V \to W$.

Given any family of formal solutions $F_t : TV \to TW$ parametrized by the points z of a disc D^k, we denote by N the total space of the bundle ν_0 normal to the homomorphism F_0. Then the isomorphism $\widetilde{F}_0 : TN \to TW$ which extends F_0 can be canonically prolongated along radii of the disc D^k to a family of isomorphisms $\widetilde{F}_t : TN \to TW$ for all $z \in D^k$, and hence the above argument applies parametrically. □

◀ Remarks

1. If the bundle $(f^*TW)/TV$ contains a trivial one-dimensional subbundle θ then it is sufficient to extend V to $V \times \mathbb{R}$. In fact, we can proceed inductively over skeleta of a triangulation of V and in this case there is no problem with the existence of θ locally over a neighborhood of a simplex.

2. Note that the the microextension trick does not work for *submersions* because the restriction of a submersion is not, in general, a submersion. In fact the h-principle is *false* for submersions of closed manifolds. ▶

Similarly one can prove the following generalization of Theorem 8.2.1 (compare 4.6.2).

8.2.2. (Immersions transversal to distribution) *Let ξ be a subbundle of TW. If $\dim V < \operatorname{codim} \xi$ then all forms of the h-principle holds for the immersions $V \to (W, \xi)$ which send V transversally to ξ.*

8.3. Sections transversal to distribution

We will need the following modification of the h-principle 7.2.1.

8.3.1. *Let $X \to V \times \mathbb{R}$ be a natural fibration and $\mathcal{R} \subset X^{(r)}$ an open differential relation invariant with respect to diffeomorphisms of the form*

$$(x,t) \mapsto (x, h(x,t)) \;\; x \in V,\, t \in \mathbb{R}.$$

Then $\mathcal{R}$ satisfies all forms of the local h-principle near $V \times 0$, and the global parametric h-principle over $V \times \mathbb{R}$.

The proof follows from the Holonomic Approximation Theorem 3.1.1 according to the same scheme as the proof of 7.2.1, with an additional remark that the perturbation h implied by 3.1.1 has special form, precisely as Proposition 8.3.1 requires.

Note that Proposition 8.3.1 is a version of the *Main Flexibility Theorem* from [**Gr86**], p. 78.

Given two manifolds X, Y and tangent distributions $\tau \subset TX$ and $\eta \subset TY$, we say that a homomorphism $TX \to TY$ maps τ transversally to η if the composition map

$$\tau \xrightarrow{F|_\tau} TY \longrightarrow TY/\eta$$

is injective when $\dim \tau + \dim \eta \leq \dim Y$, and surjective when $\dim \tau + \dim \eta \geq \dim Y$. We say that a map $f : X \to Y$ *sends τ transversally to η* if the differential $df : TX \to TY$ has the above property.

8.3.2. (Gromov [Gr86]) *Let $X \to V$ be a fibration and τ a subbundle of the tangent bundle TX. If*

$$\dim \tau + \dim V < \dim X,$$

then the sections $V \to X$ transversal to τ satisfy all forms of the h-principle.

◀ **Remarks**

1. Here the respective differential relation $\mathcal{R}$ *is not* $\operatorname{Diff} V$-invariant.

2. The condition $\dim \tau + \dim V < \dim X$ is crucial and cannot be, in general, weakened even for an open V. See, however, Theorem 14.2.1 below.

3. For a trivial fibration $V \times W \to V$ and τ equal to the vertical tangent bundle of the fibration $V \times W \to W$ Theorem 8.3.2 is just Hirsch's theorem about immersions $V \to W$, $\dim V < \dim W$ (see 8.2.1 above). ▶

Proof. of 8.3.2 Using a sufficiently small triangulation of the manifold V we can reduce the problem to its following relative version: $V = D^n$, $X = D^n \times \mathbb{R}^q$, and the section $V \to X$ is already transversal to τ near $\partial V = \partial D^n$.

A microextension trick which we are going to apply below differs from those we have used in the proof of Theorem 8.2.1: now we will extend *both* the source and the target manifolds.

Let $\mathcal{R}$ be our relation and $F : V \to X^{(1)}$ be its formal solution which is already holonomic near ∂V. Set $f = \operatorname{bs} F$. Let ξ be the subbundle of $TX|_{f(V)}$ defined by F. Consider the fibration $X \times \mathbb{R} \to V \times \mathbb{R}$ and the subbundle $\tau \times \mathbb{R} \subset T(X \times \mathbb{R})$. Let $\overline{\mathcal{R}} \subset (X \times \mathbb{R})^{(1)}$ be the differential relation which defines the sections transversal to $\tau \times \mathbb{R}$. The subbundle

$$\nu = (TX|_{f(V)})/(\tau|_{f(V)} \oplus \xi)$$

is *trivial* (this is the only place where we need the localization) and $\dim \nu \geq 1$. Therefore, we can extend F to a formal solution $\overline{F} : V \times \mathbb{R} \to X \times \mathbb{R}$ of $\overline{\mathcal{R}}$ which is holonomic near $\partial V \subset V \times \mathbb{R}$.

Let $\mathcal{A}$ be the subgroup in $\operatorname{Diff}(V \times \mathbb{R})$ which consists of diffeomorphisms fibered over V. The relation $\overline{\mathcal{R}}$ is open and $\mathcal{A}$-invariant. Therefore, according to Proposition 8.3.1 the local h-principle holds for $\overline{\mathcal{R}}$ near $V = V \times 0 \subset V \times \mathbb{R}$. The condition

$$\dim \tau + \dim V < \dim X$$

implies

$$\dim(\tau \times \mathbb{R}) + \dim(V \times \mathbb{R}) \leq \dim(X \times \mathbb{R}),$$

and hence a section $V \times \mathbb{R} \to X \times \mathbb{R}$ which is transversal to $\tau \times \mathbb{R}$ defines a section $V \to X$ which is transversal to τ. Therefore the local h-principle for $\overline{\mathcal{R}}$ implies the h-principle for $\mathcal{R}$. □

Similarly we can prove

8.3.3. *Let η be a subbundle of TV, and ξ a subbundle of TW. If* $\dim \eta < \operatorname{codim} \xi$ *then all forms of the h-principle hold for the* **maps** $(V, \tau) \to (W, \xi)$ *which send τ transversally to ξ.*

In particular,

8.3.4. *Let η be a subbundle of TV. If* $\dim \tau < \dim W$, *then the h-principle holds for "η-immersions" $f : (V, \tau) \to W$, i.e. the maps with* $\operatorname{rank} df|_\eta = \dim \eta$.

Part 3

Homotopy Principle in Symplectic Geometry

Symplectic and Contact geometry lie on the borderline between the Flexible World, governed by the laws of the *h*-principle, and the Rigid World, which deals with the differential relations for which the homotopy restrictions sufficient for the existence of formal solutions are far from being sufficient for genuine solvability. In the sixties the conjectures of V.I. Arnold, see [**Ar65, Ar78**], were directing the development of Symplectic geometry towards rigidity, while the success of symplectic applications of Gromov's *h*-principle, see [**Gr69**], put under a big question mark whether any rigid phenomena may exist in the Symplectic world, see the discussion in [**Ar86**] and historical remarks in [**Gr85**]. We will consider in the subsequent chapters the flexible side of the symplectic story, only briefly discussing some rigid phenomena in Chapter 11, and refer the readers who wish to see the rigid part of the Symplectic world to Gromov's seminal paper [**Gr85**], Arnold's paper [**Ar86**] as well as the books [**MS98**] and [**HZ94**].

Chapter 9

Symplectic and Contact Basics

This chapter contains a short introduction to Symplectic and Contact geometry. We do not pretend to be systematic. However, the chapter contains all the symplectic and contact information which is needed for our applications. We stress the similarity and the relationship between Symplectic and Complex geometries. The connection between the two geometries continues to serve as one of the most important sources of all the recent developments in Symplectic geometry after Gromov's paper [**Gr85**]. The proofs are mostly only indicated, or even omitted. The reader who is not familiar with the subject may consider the text as a long sequence of exercises, or may turn to the books [**AG90**], [**MS98**], [**HZ94**] and [**CdS01**].

9.1. Linear symplectic and complex geometries

A. Symplectic structures

A *symplectic structure*, or a *symplectic form*, on a real vector space L is a non-degenerate 2-form $\omega \in \Lambda^2 L$, i.e. skew-symmetric bilinear form $L \times L \to \mathbb{R}$. The non-degeneracy condition means that the formula

$$I_\omega(X) = \omega(X, \cdot) = X \lrcorner\, \omega\,, \quad X \in L\,,$$

defines an isomorphism $I_\omega : L \to L^*$ between L and its dual space L^*. This condition automatically implies that L is of even dimension $2n$. The non-degeneracy of ω is equivalent to the condition that $\omega^n \neq 0$, i.e. that ω^n is a volume form. The pair (L, ω) is called a *symplectic vector space*. The group of all *linear symplectomorphisms*

$$\Phi : L \to L\,, \quad \Phi^*\omega = \omega\,,$$

is denoted by $\mathrm{Sp}\,(L, \omega)$.

Any symplectic vector space (L, ω) has a *symplectic basis*

$$u_1, \dots u_n, v_1, \dots, v_n$$

such that $\omega(u_i, v_i) = 1$ and w is equal to 0 on all other pairs of basic vectors; with respect to this basis the form ω can be written as

$$\omega = \omega_0 = \sum_{i=1}^{n} p_i \wedge q_i\,, \quad p_i = \omega(u_i, \cdot)\,, \; q_i = \omega(\cdot, v_i)$$

or, equally, $\omega(X, Y) = X^T \Omega_0 Y$, where

$$\Omega_0 = \begin{bmatrix} \mathbf{0} & \mathbf{1} \\ -\mathbf{1} & \mathbf{0} \end{bmatrix}$$

In particular, as in the case of Euclidean structures, there exists, up to isomorphism, *only one* symplectic structure on L and *only one* linear symplectic group

$$\mathrm{Sp}(2n) = \mathrm{Sp}\,(\mathbb{R}^{2n}, \omega_0) \simeq \mathrm{Sp}\,(L, \omega)\,.$$

Let us point out, however, that the *space*

$$\mathcal{S}(L) = \Lambda^2 L \setminus (\Sigma = \{\omega \,|\, \omega^n = 0\}) \simeq \mathrm{GL}\,(2n, \mathbb{R})/\mathrm{Sp}\,(2n)$$

of all symplectic structures on L, in contrast to the space of Euclidean structures $\mathcal{G}(L) \simeq \mathrm{GL}\,(L)/\mathrm{O(L)}$, *is not* contractible. Note that $\mathcal{S}(L)$ consists of two identical (non-contractible) components which correspond to two orientations on L.

◀ **Exercise.** Prove that $\mathcal{S}(\mathbb{R}^4)$ is homotopy equivalent to $S^2 \sqcup S^2$. ▶

B. Symplectic orthogonal complement and classification of linear subspaces

There exists a remarkable diversity of linear subspaces of a symplectic vector space. Each linear subspace S in (L, ω) is characterized up to isomorphism by *two* numbers (s, p) where $\dim S = s$ and $\mathrm{rank}\,\omega|_S = p$. Note that p is always even. An alternative description of this classification can be given in terms of the *symplectic orthogonal complement.*

Let (L, ω) be a symplectic vector space, $\dim L = 2n$. Given a linear subspace $S \subset L$, the linear subspace

$$S^{\perp_\omega} = \{X \in L \,|\, \omega(X, Y) = 0 \;\text{ for all }\; Y \in S\}$$

is called the *symplectic orthogonal complement*, or ω-orthogonal complement of S. As in the case of the Euclidean orthogonal complement, for any $S \subset L$ we have

$$\dim S + \dim S^{\perp_\omega} = 2n \text{ and } \left(S^{\perp_\omega}\right)^{\perp_\omega} = S\,.$$

However, in general $S + S^{\perp_\omega} \neq L$. In particular, the ω-orthogonal complement of a line always contains this line, while the ω-orthogonal complement of a hyperplane is contained in the hyperplane.

A subspace $S \subset (L, \omega)$ has type (s, p) if and only if $\dim S \cap S^{\perp_\omega} = s - p$. A subspace $S \subset (L, \omega)$ is called

- *symplectic* if it has type (s, s), or equivalently if $S + S^{\perp_\omega} = L$;
- *isotropic* if it has type $(s, 0)$, i.e. $\omega|_S = 0$, or equivalently if $S \subset S^{\perp_\omega}$;
- *coisotropic* if it has type $(s, 2s - 2n)$, or equivalently if $S^{\perp_\omega} \subset S$;
- *Lagrangian* if it has type $(n, 0)$, or equivalently if $S = S^{\perp_\omega}$.

Note that the dimension of an isotropic subspace is always $\leq n$ and the dimension of a coisotropic subspace is always $\geq n$. Lagrangian subspaces can be also characterized as isotropic subspaces of maximal possible dimension, or coisotropic subspaces of the least possible dimension. The symplectic complement of an isotropic subspace is a coisotropic subspace and vice versa. The intersection $S \cap S^{\perp_\omega}$ is always isotropic. Also note that for any fixed s we have a stratification of the Grassmanian $\mathrm{Gr}_s L$ by strata which correspond to (s, p)-subspaces; the stratum of maximal dimension corresponds to the *maximal* possible p. In particular, for even s a *generic* s-dimensional linear subspace of (L, ω) is *symplectic*.

C. Complex structures

A *complex structure* on a real vector space L is an automorphism $J : L \to L$ such that $J^2 = -\mathrm{Id}_L$. This condition automatically implies that L is of even dimension $2n$. The pair (L, J) is called *complex vector space*. This definition, of course, coincides with the definition of a complex vector space as a vector space over $\mathbb{C}$; the correspondence is given by the formula $(a+ib)v = av+bJv$. The group of all linear transformations

$$\Phi : L \to L\,, \quad J = \Phi^{-1} J \Phi = J\,,$$

i.e. transformation which preserve J, is denoted by $\mathrm{GL}(L, J)$.

For any complex vector space (L, J) there exists a *J-basis*

$$u_1, \ldots, u_n, v_1, \ldots, v_n$$

such that $Ju_i = v_i$; with respect to this basis we have

$$J = J_0 = \begin{bmatrix} \mathbf{0} & -\mathbf{1} \\ \mathbf{1} & \mathbf{0} \end{bmatrix}$$

In particular there exists, up to isomorphism, *only one* complex structure on L and *only one* linear complex group

$$\mathrm{GL}(n, \mathbb{C}) = \mathrm{GL}\left(\mathbb{R}^{2n}, J_0\right) \simeq \mathrm{GL}\left(L, J\right).$$

As we will see below, the space

$$\mathcal{J}(L) \simeq \mathrm{GL}\left(2n, \mathbb{R}\right)/\mathrm{GL}\left(n, \mathbb{C}\right)$$

of all complex structures on L is homotopy equivalent to the space $\mathcal{S}(L)$ of symplectic structures on L. In particular, $J(L)$ consists of two identical (non-contractible) components which correspond to two orientations of L.

◀ **Exercise.** Prove that $\mathcal{J}(\mathbb{R}^4)$ is homotopy equivalent to $S^2 \sqcup S^2$. ▶

D. Classification of real linear subspaces in a complex vector space

Each linear subspace S in (L, J) is characterized up to isomorphism by *two* numbers (s, p) where $\dim S = s$ and $p = \dim\left(S \cap JS\right)$. Note that $S \cap JS$ is J-invariant and p is always even.

A subspace $S \subset (L, J)$ is called

- *complex* if it has type (s, s), or equivalently if $S = JS$; in other words a subspace is complex if it is invariant with respect to J;
- *real* or *totally real* if it has type $(s, 0)$, or equivalently if $S \cap JS = 0$; in other words a subspace is totally real if it contains no complex subspaces of positive dimension;
- *co-real* if it has type $(s, 2s - 2n)$, or equivalently $S + JS = L$.

Note that the dimension of a real subspace is always $\leq n$, while the dimension of a co-real subspace is always $\geq n$. The intersection $S \cap JS$ is J-invariant and hence always complex. Also note that for any fixed s we have a stratification of the Grassmanian $\mathrm{Gr}_s L$ by the strata which correspond to (s, p)-subspaces; the stratum of maximal dimension corresponds to the *minimal* possible p. In particular, a *generic* s-dimensional linear subspace of (L, ω) is *real* if $s \leq n$ and *co-real* if $s \geq n$. The odd-dimensional subspaces of type $(s, s - 1)$ are called CR-subspaces, where the notation CR stands for Cauchy-Riemann. Any real hyperplane in a complex space is automatically a CR-subspace.

E. Hermitian structures and homotopy equivalence $\mathcal{J}(L) \sim \mathcal{S}(L)$

A *Hermitian structure* on a (real) vector space L is a pair $H = (J, \omega)$ where J and ω are *compatible* complex and symplectic structures on L, i.e.

- ω is J-invariant, i.e. $\omega(JX, JY) = \omega(X, Y)$ for all $X, Y \in L$;
- ω is J-positive, i.e. $\omega(X, JX) > 0$ for all $X \in L$.

This definition is equivalent to the standard definition of a Hermitian structures on L as a *positive definite* Hermitian form $H(X, Y)$ on the complex vector space (L, J); the correspondence is given by the formula

$$H(X, Y) = \omega(X, JY) - i\omega(X, Y)\,.$$

A linear transformation $L \to L$ which preserves H is called *unitary*, and the group of all unitary transformations is denoted by $\mathrm{U}\,(L, H)$.

For any Hermitian vector space (L, H) there exists a basis

$$u_1, \ldots, u_n, v_1, \ldots, v_n$$

which is simultaneously J-basis and symplectic basis. With respect to this basis we have $H = H_0 = (J_0, \omega_0)$. In particular, as in the case of Euclidean spaces there exists, up to isomorphism, *only one* Hermitian structure on L and *only one* unitary group

$$\mathrm{U}(n) = \mathrm{U}\,(\mathbb{R}^{2n}, H_0) \simeq \mathrm{U}\,(L, H)\,.$$

The *space* $\mathcal{H}(L)$ of all Hermitian structures on L is a subspace of the product $\mathcal{J}(L) \times \mathcal{S}(L)$ and thus we have natural projections $p_{\mathcal{J}} : \mathcal{H} \to \mathcal{J}$ and $p_{\mathcal{S}} : \mathcal{H} \to \mathcal{S}$.

9.1.1. (Homotopy equivalence $\mathcal{J}(L) \sim \mathcal{S}(L)$) *The projections $p_{\mathcal{J}} : \mathcal{H} \to \mathcal{J}$ and $p_{\mathcal{S}} : \mathcal{H} \to \mathcal{S}$ are surjective maps. Both maps $p_{\mathcal{J}}$ and $p_{\mathcal{S}}$ are fibrations with contractible fibers. Moreover, both $p_{\mathcal{J}}$ and $p_{\mathcal{S}}$ are homotopy equivalences and hence there exist a canonical homotopy equivalence $\mathcal{J}(L) \sim \mathcal{S}(L)$.*

Proof. The surjectivity of $p_{\mathcal{J}}$ and $p_{\mathcal{S}}$ is evident. Each fiber

$$p_{\mathcal{J}}^{-1}(J) \simeq p_{\mathcal{J}}^{-1}(J_0) = \mathrm{GL}(n, \mathbb{C})/\mathrm{U}(n)$$

is a convex subset of a vector space and hence is contractible. The convexity of $p_{\mathcal{J}}^{-1}(J)$ also implies that $p_{\mathcal{J}}$ is a homotopy equivalence. Each fiber

$$p_{\mathcal{S}}^{-1}(\omega) \simeq p_{\mathcal{S}}^{-1}(\omega_0) = \mathrm{Sp}(2n)/\mathrm{U}(n)$$

is contractible, as can be seen from the polar decomposition (see, for example, [**MS98**]). Therefore, there exists a section $f : \mathcal{S} \to \mathcal{H}$ of the fibration $p_\omega : \mathcal{H} \to \mathcal{S}$; using the fiberwise polar decompositions with respect to the metrics $g_\omega(X, Y) = \omega(X, JY)$ where $(J, \omega) = f(\omega)$ we can realize the contraction simultaneously in all fibers. Thus, the projection $p_{\mathcal{S}}$ is a homotopy equivalence. □

◀ **Remarks**

1. The space $\mathcal{H}_\omega = p_{\mathcal{S}}^{-1}(\omega)$ admits another interpretation which manifests its contractibility. Namely, $\mathcal{H}_\omega$ can be identified with the so-called Siegel upper-half space, i.e. the space of matrices of the form $A + iB$, where A, B are real symmetric $n \times n$-matrices, and B is positive definite, see [**Si64**] for the details.

2. Instead of the space $\mathcal{H}$ one can consider a bigger space $\widetilde{\mathcal{H}}$ of the pairs (J, ω) with only positivity condition

$$\omega(X, JX) > 0 \text{ for all } X \in L\,.$$

A theorem similar to 9.1.1 is valid also for $\widetilde{H}$ and the projections $\widetilde{p}_{\mathcal{J}} : \widetilde{\mathcal{H}} \to \mathcal{J}$ and $\widetilde{p}_{\mathcal{S}} : \widetilde{\mathcal{H}} \to \mathcal{S}$. ▶

9.2. Symplectic and complex manifolds

A. Symplectic and complex vector bundles

Using the respective linear notions one can define *symplectic, complex and Hermitian vector bundles.* For example, a *symplectic vector bundle* (X, ω) over a manifold V is a real vector bundle $p : X \to V$ equipped with a symplectic form ω_v on each fiber $X_v = p^{-1}(v)$ which smoothly depends on $v \in V$. Equivalently the symplectic structure on a real vector bundle $X \to V$ can be defined as a section

$$V \to \Lambda^2 X \setminus \Sigma\,,$$

where $\Sigma \cap \Lambda^2 X_v = \{\omega \,|\, \omega^n = 0\}$.

Two symplectic structures ω_0, ω_1 on a real vector bundle X are homotopic if they can be connected by a family $\{\omega_t\}_{t \in I}$ of symplectic structures. As follows from Proposition 9.1.1, there is no difference from the homotopy point of view between symplectic, complex and Hermitian vector bundles, i.e. there exists a canonical one-to-one correspondence between the homotopy classes of symplectic and complex structures on a given real vector bundle X and, moreover, a canonical homotopy equivalence

$$\mathcal{S}(X) \sim \mathcal{H}(X) \sim \mathcal{J}(X)\,,$$

where $\mathcal{S}(X)$, $\mathcal{J}(X)$ and $\mathcal{H}(X)$ are the spaces of symplectic, complex and Hermitian structures on X.

◀ **Remark.** For a given symplectic structure ω on a vector bundle X, the connected component $\mathcal{J}_\omega \subset \mathcal{J}(X)$ which corresponds to the connected component of $\mathcal{S}_\omega \in \mathcal{S}(X)$ consist of all complex structures $J \in \mathcal{J}(X)$ such that J is fiberwise compatible with a symplectic structure $\omega' \in \mathcal{S}_\omega$ (and vice versa). ▶

B. Almost symplectic and almost complex manifolds

An *almost symplectic* (resp. *almost complex, Hermitian*)) structure on an even-dimensional manifold V is a symplectic (resp. complex, Hermitian) structure on the tangent bundle TV. Equivalently, an almost symplectic structure on V is a non-degenerate differential 2-form ω on V. All these structures have local invariants and in the almost complex case were intensively studied, similarly to the case of Riemannian metrics. As far as we know the differential geometry of almost symplectic structures is practically non-existent.

C. Submanifolds of almost symplectic and almost complex manifolds

Using the respective linear notions one can define (s,p)-*submanifolds* of almost symplectic (resp. almost complex) manifolds, and in particular *isotropic, co-isotropic, Lagrangian, almost symplectic* (resp. *totally real, co-real, almost complex*) submanifolds. Note that isotropic submanifolds $S \subset V$ are characterized by the condition $\omega|_S \equiv 0$. The isotropic submanifolds of dimension $< \frac{1}{2}\dim V$, i.e. non-Lagrangian isotropic submanifolds, are called *subcritical.*

D. Symplectic and complex manifolds: infinitesimal description

An almost symplectic structure ω is called *integrable* if $d\omega = 0$, i.e. ω is a closed two-form. Such a differential form is called *symplectic form.* An almost complex structure J on V is called *integrable* if the *Nijenhuis tensor*

$$N_J(X,Y) = [JX,JY] - J[JX,Y] - J[X,JY] - [X,Y]$$

vanishes. A Hermitian structure $H = (J,\omega)$ on V is called *integrable* if both J and ω are integrable; this is equivalent to the equality $\nabla_g J = 0$ where ∇_g is the covariant derivative with respect to the metric $g(X,Y) = \omega(X,JY)$. A manifold V provided with an integrable almost symplectic (resp. almost complex, Hermitian) structures is called *symplectic* (resp. *complex, Kähler*) manifold. A Kähler manifold can equivalently be defined as a *complex* manifold provided with a positive definite Hermitian form H such that the imaginary part of H is closed; such a form is called *Kähler metric.* An *almost complex* manifold V provided with a Kähler metric is called *almost Kähler* manifold.

E. Symplectic and complex manifolds: local description

According to a theorem of Nirenberg-Newlander (see [**NN57**]), any sufficiently smooth integrable almost complex structure is locally equivalent to the standard complex structure on $\mathbb{C}^n$. Thus, equivalently, a complex manifold can be characterized by the existence of local charts $\{\mathrm{U}_i \to (\mathbb{R}^{2n}, J_0) = \mathbb{C}^n\}$ glued together by holomorphic maps (this is the standard definition of a complex manifold).

For symplectic manifolds we have a similar situation. According to Darboux' theorem (see 9.3.2 below) any symplectic form is locally equivalent to the *standard symplectic form* $\omega_0 = \sum\limits_1^n dx_i \wedge dy_i$ on $\mathbb{R}^{2n}$. Thus, equivalently, a symplectic manifold can be characterized by the existence of local *Darboux charts* glued together by *symplectomorphisms*, i.e. diffeomorphisms which preserve the canonical form.

Note that for 2-dimensional manifolds we have

almost symplectic structure=symplectic structure=area form ;

almost complex structure=complex structure=conformal structure .

F. Submanifolds of symplectic and complex manifolds

Note that a symplectic (complex) *submanifold* $S \subset V$ of a symplectic (complex) manifold V inherits an *integrable* symplectic (complex) structure. This contrasts with the Riemannian case: the symplectic integrability condition is analogous to the condition in the Riemannian case of being locally Euclidean, but submanifolds of locally Euclidean manifolds need not, of course, to be locally Euclidean.

Let us mention the following property of (s,p)- submanifolds of (integrable) symplectic manifolds (see, for example, [**MS98**]).

9.2.1. *For any (s,p)-submanifold $S \subset V$ of a symplectic manifold V the $(s-p)$-dimensional distribution $TS \cap (TS)^{\perp_\omega} \subset TS$ is integrable. The corresponding foliation on S, called* **characteristic***, consists of isotropic leaves. In particular, any s-dimensional coisotropic submanifold $S \subset V$ carries a canonical $(2n-s)$-dimensional characteristic isotropic foliation.*

Note that any hypersurface $S \subset V$ is always coisotropic and hence carries a canonical one-dimensional characteristic foliation.

Complex tangent subspaces $TS \cap J(TS)$ to a real hypersurface S of an almost complex manifold V form a complex tangent distribution ξ of real codimension 1, which is called a CR-*structure* on S. A coorientation ν of S in V defines a coorientation $J\nu$ of ξ in S. Choose a 1-form α such that

$\xi = \operatorname{Ker} \alpha$ and $\alpha(J\nu) > 0$. It is straightforward to check that if the structure J is integrable, then the formula

$$L(X, Y) = d\alpha(X, JY) - id\alpha(X, Y), \quad X, Y \in \xi\,,$$

defines an Hermitian form on S, called the *Levi form*. The Levi form is defined up to multiplication by a positive function. The cooriented hypersurface S is called *strictly pseudo-convex* if the form L is positive definite.

G. Examples of symplectic manifolds

1. *Cotangent bundle*

An important example of a symplectic manifold is provided by the cotangent bundle T^*M of a smooth manifold M. The symplectic form ω on T^*M is the differential of the famous canonical 1-form $p\,dq$. In coordinate notations the symplectic structure on T^*M can be described as follows. If $M = \mathbb{R}^n$ then $\mathbb{R}^{2n} = T^*\mathbb{R}^n$ is endowed with the canonical symplectic structure

$$\omega_0 = d(p\,dq) = \sum_1^k dp_i \wedge dq_i\,,$$

where the coordinates $q = (q_1, \ldots, q_n)$ and $p = (p_1, \ldots, p_n)$ are chosen in such a way that the projection $T^*\mathbb{R}^n \to \mathbb{R}^n$ is given by $(p, q) \mapsto q$. Let us observe that any diffeomorphism $f : \mathbb{R}^n \to \mathbb{R}^n$ lifts to a symplectomorphism $f_* : T^*\mathbb{R}^n \to T^*\mathbb{R}^n$ by the formula

$$f_*(p, q) = (f(q), (df^*)^{-1}(p))\,.$$

Thus a coordinate atlas $M = \bigcup\limits_j U_j$ on M lifts to a symplectic atlas

$$T^*M = \bigcup_j T^*U_j$$

with gluing symplectomorphisms lifted by the above formula.

2. *Symplectic stucture on Kähler manifolds*

Kähler geometry serves as a rich source of examples of symplectic manifolds. The imaginary part of a Kähler metric is a symplectic form, and hence any Kähler manifold is automatically symplectic. Moreover, complex submanifolds of Kähler manifolds are Kähler, and hence symplectic. Complex affine space $\mathbb{C}^n$ and the complex projective space $\mathbb{C}P^n$ have canonical Kähler metrics ($\sum_{i=1}^n z_i \bar{z}_i$ on $\mathbb{C}^n$, and the *Fubini-Study* metric on $\mathbb{C}P^n$) and hence any *affine* complex manifold, i.e. a complex submanifold of $\mathbb{C}^n$, and any *projective* complex manifold, i.e. a complex submanifold of the complex projective space $\mathbb{C}P^n$, are symplectic.

The above examples of complex analytic origin can in a certain sense be reversed. Note that according to Proposition 9.1.1 the space $\mathcal{J}_\omega(W)$ of all *almost complex structures* $J : TW \to TW$ compatible with $\omega|_{T_W}$ on every tangent space T_xW, $x \in W$, of a symplectic manifold (W, ω) is non-empty and contractible. In particular, any symplectic manifold (M, ω) admits a compatible almost complex structure J, and thus (W, ω, J) is an *almost Kähler manifold.*

3. *Symplectic bundle over a symplectic manifold*

The following observation belongs to W.P. Thurston (see [**Th76**]):

9.2.2. *Let (V, ω) be a symplectic manifold and $\pi : X \to V$ a symplectic vector bundle. Then there exists a symplectic structure $\widetilde{\omega}$ on a neighborhood $\mathcal{O}p\, V$ of the 0-section $V \subset X$ such that $\widetilde{\omega}|_V = \omega_V$ and the restriction of $\widetilde{\omega}$ to the fibers of the fibration X defines their linear symplectic structure.*

Proof. By definition the total space X of a symplectic vector bundle admits a closed form η such that its restriction to the fibers of the fibration defines there the linear symplectic structure, and the restriction of η to the 0-section V vanishes. Then the form $\eta + \pi^*\omega_V$ is symplectic on a neighborhood $\mathcal{O}p\, V$ of the 0-section and has the required properties. □

H. Embeddings and immersions into (almost) symplectic and (almost) complex manifold.

In an obvious way we define Lagrangian, isotropic and coisotropic immersions into (almost) symplectic manifolds and totally real and co-real immersions into (almost) complex manifolds.

When considering (almost) symplectic immersions into an (almost) symplectic manifold (W, ω_W) we should differentiate between immersions which induce on the source an (almost) symplectic structure, and the immersions of another symplectic manifold which induce on it the (almost) symplectic structure which was a priori given. The same way we should treat *(almost) complex* immersions into an (almost) complex manifold.

Let (W, ω_W) be a symplectic manifold. A map $f : V \to (W, \omega_W)$ is called *symplectic* if the form $f^*\omega_W$ is non-degenerate (and hence symplectic). Such a map is automatically an immersion. If V is already endowed with another symplectic structure ω_V then we call a map $f : (V, \omega_V) \to (W, \omega_W)$ *isometric symplectic* or *isosymplectic* if $f^*\omega_W = \omega_V$.

The same way one can define *complex* and *isocomplex* immersions into a complex manifold.

◀ **Exercise.** Define all kinds of the respective "formal" immersions (Lagrangian, symplectic, etc.) and the corresponding differential relations. Which of these relations are open? Invariant with respect to Diff V? ▶

9.3. Symplectic stability

By symplectic stability we mean the absence of non-trivial local invariants for objects related to *integrable* symplectic structures. Here the locality refers both to the manifold itself and to the space of symplectic structures.

For a smooth family of differential p-forms ω_t on a manifold W the *time-derivative*

$$\dot{\omega}_t = \frac{d}{dt}\,\omega_t$$

is again a family of p-forms. Note that the time-derivative commutes with the exterior differentiation:

$$d(\dot{\omega}_t) = (\dot{d\omega_t})\,.$$

For a (time-dependent) vector field v_t on W denote by $\varphi_t = e^{tv_t}$ the phase flow on W generated by v_t, i.e. φ_t is determined by the differential equation

$$\dot{\varphi}_t(x) = v_t(\varphi_t(x))\,, \quad \varphi_0(x) = x\,.$$

The isotopy (flow) φ_t is always defined at least locally.

Given a differential p-form ω and a vector field X on W, the derivative

$$\mathcal{L}_X\omega = \dot{\omega}_t|_{t=0}\,, \quad \omega_t = (e^{tX})^*\omega\,,$$

is called *Lie derivative of the form ω along X*. Let us recall E. Cartan's formula:

$$\mathcal{L}_X\omega = X \lrcorner\, d\omega + d(X \lrcorner\, \omega)\,.$$

Suppose we have a *homotopy* ω_t, $t \in [0,1]$, of differential p-forms on W; can we realize this homotopy by an *isotopy* $\varphi_t : W \to W$ such that $\varphi_t^*\omega_0 = \omega_t$? It is sufficient to find a corresponding time-dependent vector field v_t such that $\varphi_t = e^{tv_t}$. Differentiation of the equation $\varphi_t^*\omega_0 = \omega_t$ with respect to t gives us the equation

$$\mathcal{L}_{v_t}\omega_t = \dot{\omega}_t \ \text{ for all } \ t \in [0,1]$$

with respect to v_t.

The following proposition is a starting point for the symplectic stability results.

9.3.1. (Solution to the equation $\mathcal{L}_{v_t}\omega_t = \dot\omega_t$ for an exact homotopy of symplectic forms) *Let $\omega_t = \omega_0 + d\alpha_t$ be a smooth family of symplectic forms on a manifold W and $v_t = I^{-1}_{\omega_t}(\dot\alpha_t)$ (vector field ω_t-dual to the 1-form $\dot\alpha_t$). Then*

$$\mathcal{L}_{v_t}\omega_t = \dot\omega_t \ \text{ for all } \ t \in [0,1]\,.$$

Indeed, we have

$$\mathcal{L}_{v_t}\omega_t = d(v_t \lrcorner\, \omega_t) = d(\dot\alpha_t) = (\dot{\overline{d\alpha_t}}) = \dot\omega_t\,.$$

Here are few remarkable corollaries of this proposition:

9.3.2. (Stability Theorems)

(Stability near a compact set) *Let $A \subset W$ be a compact subset. Let $\omega_t = \omega_0 + d\alpha_t$ be a family of symplectic forms on $\mathcal{O}p\, A \subset W$ such that $\alpha_t|_{TW|_A} = 0$. Then there exists an isotopy $\varphi_t : \mathcal{O}p\, A \to W$, fixed on A, such that $\varphi_t^*\omega_0 = \omega_t$.*

(Darboux' Theorem) *Any symplectic form is locally equivalent to the form $\omega_0 = \sum\limits_1^n dx_i \wedge dy_i$ on $\mathbb{R}^{2n} = T^*\mathbb{R}^n$.*

(Moser's Theorem) *Let $\omega_t = \omega_0 + d\alpha_t$ be a family of symplectic forms on a closed manifold W. Then there exists a canonical isotopy $\varphi_t : W \to W$ such that $\omega_t = \varphi_t^*\omega$.*

(Relative Moser's Theorem) *Let ω_t be a family of symplectic forms on a compact manifold W with boundary such that $\omega_t = \omega_0$ over $\mathcal{O}p\, \partial W$ and the relative cohomology class $[\omega_t - \omega_0] \in H^2(W, \partial W)$ vanishes for all $t \in [0,1]$. Then there exists an isotopy $\varphi_t : W \to W$ which is fixed on $\mathcal{O}p\, \partial W$ and such that $\varphi_t^*(\omega_0) = \omega_t\,,\ \ t \in [0,1]$.*

(Weinstein's Theorem) *Any isotropic and, in particular, Lagrangian immersion $L \to W$ extends to an isosymplectic immersion $U \to W$, where U is a neighborhood of the zero-section in the cotangent bundle T^*L endowed with its canonical symplectic structure.*

(Symplectic Neighborhood Theorem) *Let $f : (V, \omega_V) \to (W, \omega_W)$ be an isosymplectic immersion and $E \to V$ be the symplectic vector bundle whose fiber over a point $v \in V$ is the space $(df(T_vV))^{\perp_\omega}$ ω_W-dual to $df(T_vV) \subset T_{f(v)}W$. Then f extends to an isosymplectic immersion*

$$\widehat{f} : (\mathcal{O}p\, V, \omega_E) \to (W, \omega_W)\,,$$

where $\mathcal{O}p\, V$ is a neighborhood of the 0-section V in the total space of the symplectic vector bundle $E \to V$, and ω_E is the symplectic form on $\mathcal{O}p\, V \subset E$ constructed in Lemma 9.2.2 above.

◀ **Remark.** All the above results hold in parametric form. It is important, however, to keep in mind that in the parametric version of the *Symplectic neighborhood theorem* the symplectic bundle E also varies with the parameter, and even though all of them are equivalent there could be a homotopical obstruction if one tries to find a uniformization by a fixed symplectic model. ▶

Proof.

Stability near a compact set: The isotopy

$$\varphi_t = e^{tv_t} : \mathcal{O}p\ A \to \mathcal{O}p\ A$$

generated by the vector field $v_t = I_{\omega_t}^{-1}(\dot{\alpha}_t)$ is fixed on A and thus is defined on $\mathcal{O}p\,A$ for all $t \in [0,1]$.

Darboux: All linear symplectic forms are equivalent. Hence, one can assume that ω and ω_0 coincide at the origin. Then the linear homotopy

$$\omega_t = (1-t)\omega_0 + t\omega$$

consists of symplectic forms on $\mathcal{O}p\,0$. Moreover, $\omega_t = \omega_0 + d\alpha_t$, where $\alpha_t(0) = 0$ for all $t \in [0,1]$. Hence, we can apply *Stability near* $A = \{0\}$.

Moser: The vector field $v_t = I_{\omega_t}^{-1}(\dot{\alpha}_t)$ here integrates on the whole manifold W to an isotopy φ_t such that $\varphi_t^*\omega_0 = \omega_t$ for all $t \in [0,1]$.

Relative Moser: The proof is the same as in the absolute case with an additional remark that the forms α_t can be chosen equal to 0 on $\mathcal{O}p\,\partial W$, and then $v_t|_{\mathcal{O}p\,\partial W} = 0$ as well.

Weinstein: Any isotropic submanifold L of dimension k admits a transversal isotropic k-plane field θ such that the bundle $TL \oplus \theta$ is symplectic with respect to the symplectic form induced by ω. Moreover the space of isotropic subspaces of $\mathbb{R}^{2n}$ which trivially intersect a fixed isotropic subspace L and form with it a symplectic subspace of $\mathbb{R}^{2n}$ is contractible, and hence the space of isotropic plane fields θ which satisfies the above property is contractible as well. The immersion $L \to W$ extends in a homotopically unique way to an immersion $h : \mathcal{O}p\,L \to W$ which is isosymplectic along L with respect to the standard symplectic structure ω_0 on T^*L and symplectic structure ω on W, where $\mathcal{O}p\,L$ is a neighborhood of L in T^*L. In particular the symplectic forms ω_0 and $\widetilde{\omega} = h^*\omega$ on $\mathcal{O}p\,L$ coincide along L, and hence $\widetilde{\omega} = \omega_0 + d\alpha$ where the 1-form α can be chosen to be equal to 0 on L. Hence we can apply *Stability near* L.

Symplectic Neighborhood: By the definition of the bundle $E \to V$ the immersion f extends to an immersion $\widetilde{f} : \mathcal{O}p\,V \to W$ such that $\widetilde{f}^*(\omega_W)$ coincides with ω_E on $TE|_V$. Hence $\widetilde{f}^*(\omega_W) - \omega_E = d\alpha$ where the 1-form

α vanishes on $TE|_V$, and therefore we can construct the required isotopy using *Stability near* V. ☐

9.4. Contact manifolds

A. Contact forms and contact structures on manifolds

A 1-*form* α on a $(2n+1)$-dimensional manifold V is called *contact* if the restriction of $d\alpha$ to the $(2n)$-dimensional tangent distribution $\xi_\alpha = \operatorname{Ker} \alpha$ is non-degenerate (and hence symplectic). Equivalently, we can say that a 1-form α is contact if $\alpha \wedge (d\alpha)^n$ does not vanish on V. A codimension 1 *tangent distribution* ξ on V is called a *contact structure* or a *contact distribution* if it can be locally (and in the coorientable case globally) defined by the Pfaffian equation $\alpha = 0$ for some choice of a contact form α. The pair (V, ξ) in this case is called a *contact manifold.*

◀ **Example.** The *standard contact structure* ξ_0 on $\mathbb{R}^{2n+1}$ is defined by the Darboux contact 1-form

$$\alpha_0 = dz - \sum_1^n y_i dx_i$$

in the coordinates $(x_1, \ldots, x_{n-1}, y_1, \ldots, y_{n-1}, z)$ (see Fig.9.1). ▶

Figure 9.1. The standard contact structure $dz - ydx = 0$ on $\mathbb{R}^3$

B. Nonexistence of local contact geometry

Similar to the symplectic case, contact manifolds have no local geometry: according to a contact version of Darboux' theorem (see 9.5.2 below), any $(2n+1)$-dimensional contact manifold is locally isomorphic to the standard contact $\mathbb{R}^{2n+1}$. Thus, equivalently, any contact manifold can be defined by a contact atlas which consists of Darboux charts glued by contactomorphisms, i.e. diffeomorphisms preserving the standard contact structure ξ_0 (but not necessarily the contact form α_0!). The contact *forms* have no local invariant either: any contact form on a $(2n+1)$-dimensional manifold is locally isomorphic to the standard contact form α_0 on $\mathbb{R}^{2n+1}$ (see 9.5.2 below).

C. Orientations and conformal class $\mathrm{CS}(\xi)$ associated to a contact structure ξ

Any contact *form* α on $V = V^{2n+1}$ defines an orientation of V by the volume form $\alpha \wedge (d\alpha)^n$, an orientation of the distribution $\xi = \operatorname{Ker} \alpha$ by the form $(d\alpha)^n$ and (of course) a coorientation of ξ. Some of these orientation survive when we pass to contact *structures*: any contact *structure* ξ defines an orientation of the $(2n+1)$-dimensional manifold V if n is odd and an orientation of ξ if n is even.

The symplectic structure $d\alpha|_\xi$ on $\xi = \operatorname{Ker} \alpha$ almost survives when we pass from a contact form α to the contact structures $\xi = \operatorname{Ker} \alpha$ which this form defines: the *conformal class* of $d\alpha|_\xi$ depends only on ξ (because $d(f\alpha)|_\xi = f d\alpha|_\xi$ for a function $f : V \to \mathbb{R}$). We denote this class by $\mathrm{CS}(\xi)$.

In a more invariant fashion we can canonically associate with any codimension one tangent distribution $\xi \subset TV$ a defining 1-form α valued in the line bundle $\lambda = TV/\xi$. The differential of this form restricted to ξ is a 2-form on ξ valued in λ. Its value on vectors $X, Y \in \xi$ is defined by first extending X, Y locally to vector fields $\widetilde{X}, \widetilde{Y}$ tangent to ξ, then taking their Lie bracket with the sign minus: $-[X, Y] = YX - XY$, and then projecting it to $\lambda = TV/\xi$. If ξ is a contact structure, then the form ω is symplectic. Note that in this definition ω depends only on ξ. A choice of an $\mathbb{R}$-valued contact form trivializes (and, in particular, normalizes) λ and thus allows us to treat ω as a usual $\mathbb{R}$-valued symplectic form on ξ which is, as a trade-off, only a *conformal* invariant of ξ.

Given a *cooriented* contact structure $\xi_+ = \operatorname{Ker} \alpha$, the *positive-conformal* class of the symplectic structure $d\alpha|_\xi$ depends only on ξ_+; we denote this class by $\mathrm{CS}(\xi_+)$.

D. Integral submanifolds of a contact distribution

According to Frobenius' theorem, the contact condition is a condition of maximal non-integrability of the tangent hyperplane field ξ. In particular, all integral submanifolds of ξ have dimension $\leq n$. On the other hand, n-dimensional integral submanifolds, called *Legendrian*, always exist in abundance, see Section 9.6 below. Integral submanifolds of dimension $< n$ are called *subcritical.*

E. Reeb vector field of a contact form

Any contact form α on V defines a *one-dimensional* distribution

$$\operatorname{Ker} d\alpha \subset TV$$

and hence a one-dimensional foliation $\mathcal{R}_\alpha$ on V, which is called the *Reeb foliation.* Note that $\mathcal{R}_\alpha$ is transversal to the contact distribution $\xi_\alpha = \operatorname{Ker}\alpha$. The condition $\alpha(X) = 1$ defines a unique vector field $X = R_\alpha$ tangent to $\mathcal{R}_\alpha$; this vector field is called *Reeb vector field.* The flow of R_α preserves the contact form α.

The Reeb foliation $\mathcal{R}_\alpha$ does not survive when we pass from a contact form α to the underlying contact structure $\xi = \operatorname{Ker}\alpha$; however, some of its *invariants* do, see [**EGH00**].

F. Examples of contact manifolds.

1. *Contactization of the cotangent bundle*

The canonical contact structure on the 1-jet space $J^1(M,\mathbb{R}) = T^*(M)\times\mathbb{R}$ is defined by the contact form $dz - p\,dq$ on $T^*(M)\times\mathbb{R}$, where the coordinate z corresponds to the second factor, and where we identify the form $p\,dq$ on T^*M with its pull-back to $J^1(M,\mathbb{R})$. For $M = \mathbb{R}^n$ this structure coincide with the standard contact structure on $\mathbb{R}^{2n+1}$.

2. *Contactization of an exact symplectic manifold*

Similarly to the above *contactization* of the cotangent bundle $(T^*M, d(p\,dq))$ any exact symplectic manifold $(W, d\alpha)$ can be contactized to a contact manifold

$$(V = W\times\mathbb{R}\,,\ \xi = \operatorname{Ker}(dz-\alpha))\,,$$

or to a contact manifold

$$(V' = W\times(\mathbb{R}/\mathbb{Z})\,,\ \xi = \operatorname{Ker}(dz-\alpha))\,.$$

3. *Pre-quantization of an integral symplectic manifold*

The latter "cyclic" form of the contactization (or, as it is otherwise called, *pre-quantization*) construction can be generalized to a symplectic manifold (W,ω) whose form is not exact but *integral*, i.e. which belongs to a cohomology class from $H^2(W;\mathbb{Z})$. In this case the circle bundle $V'\to W$ with the first Chern class equal $[\omega]$ admits a S^1-connection whose curvature is equal to ω. This connection, viewed as a family of horizontal planes, represents an S^1-invariant contact structure ξ on V'.

4. *Space of contact elements of a smooth manifold*

A point of the projectivized cotangent bundle PT^*M is a tangent hyperplane to V which can be identified with a line in T^*M. The canonical 1-form $p\,dq$ does not descend to PT^*M, but its kernel does and thus it defines a canonical contact structure on PT^*M. This contact structure is not coorientable. The double cover of PT^*M, which carries a coorientable contact structure, is the associated spherical bundle ST^*M which can be viewed as the space of co-oriented tangent hyperplanes (co-oriented contact elements).

5. *Strictly pseudo-convex hypersurfaces*

As we already explained above in Section 9.2, complex geometry serves as a rich source of examples of *symplectic* manifolds. It is a rich source of examples of *contact* manifolds as well, because the CR-structure $\xi = TS \cap JTS$ on a strictly pseudo-convex hypersurface S in a complex manifold V is a contact structure on S. In particular, the standard sphere $S^{2n-1} \subset \mathbb{C}^n$ carries a canonical contact structure.

As in the symplectic case, the above example of complex analytic origin can in a certain sense be reversed. If (V, ξ) is a contact manifold, then we can endow the bundle ξ in a homotopically unique way with a complex structure J compatible with ξ in the following sense: the Hermitian (Levi) form $\omega(X, JY) - i\omega(X, Y)$, where $\omega \in CS(\xi)$ is positive definite. In other words, any contact manifold can be viewed as a strictly pseudo-convex but not necessarily integrable CR-manifold, or as a strictly pseudo-convex hypersurface in an almost complex manifold. Note that if $\dim V = 3$ then the CR-structure can be chosen integrable, see [**El85**].

6. *Symplectic bundle over a contact manifold*

The following observation is the contact analog of Lemma 9.2.2.

9.4.1. *Let (V, ξ) be a contact manifold and $\pi : E \to V$ a symplectic vector bundle. Then there exists a contact structure $\widetilde{\xi}$ on a neighborhood U of the 0-section V such that $\widetilde{\xi}|_V = \xi$, $\widetilde{\xi}$ is tangent to the fibers of the fibration E viewed as a subbundle of $T(E)|_V$, and the given symplectic structure on these fibers belong to the the conformal class* $\mathrm{CS}(\widetilde{\xi})$.

Proof. We will consider here only the case of a cooriented ξ. Let α be the contact form which defines ξ, and η a closed form on E such that

- its restriction to the fibers of the fibration defines there the given linear symplectic structure;
- the restriction of η to the 0-section V vanishes.

Then the form η is exact and by the Poincaré lemma there exists a primitive β of η such that $\beta|_{TE|_V} = 0$. The form $\beta + \pi^*\alpha$ defines on a neighborhood $\mathcal{O}p\, V$ a contact structure $\widetilde{\xi}$ with the required properties. □

H. Symplectization and projectivization

Symplectization

Let (V, ξ) be a $(2n-1)$-dimensional contact manifold. The $2n$-dimensional manifold $\widehat{V} = (TV/\xi)^* \setminus V$, called the *symplectization* of (V, ξ), carries a natural symplectic structure ω induced by a tautological embedding $\widehat{V} \to T^*V$ which assigns to each linear form

$$TV/\xi \to \mathbb{R}$$

the corresponding form

$$TV \to TV/\xi \to \mathbb{R}\,.$$

Moreover, the symplectization $\widehat{V}$ carries a canonical 1-form $\widehat{\alpha}_\xi$ induced from the canonical 1-form $p\,dq$ on T^*V by this embedding. A choice of a contact form α (if ξ is coorientable) defines a section $V \to \widehat{V}$ and, in particular, a splitting

$$\widehat{V} = V \times (\mathbb{R} \setminus 0)\,.$$

In this case we can choose the positive half $V \times \mathbb{R}_+$ of $\widehat{V}$ and call it symplectization as well. The symplectic structure ω can be written in terms of this splitting as $d(\tau\alpha)$, $\tau > 0$. Note that the vector field $T = \tau\frac{\partial}{\partial\tau}$ is *conformally symplectic* or, as it is also called, *Liouville*: we have

$$\mathcal{L}_T\omega = \omega \ \text{ as well as } \ \mathcal{L}_T(\tau\alpha) = \tau\alpha\,.$$

All the notions of contact geometry can be formulated as the corresponding symplectic notions, invariant or equivariant with respect to this conformal action. For instance, any contact diffeomorphism of V lifts to an equivariant symplectomorphism of $\widehat{V}$; Legendrian submanifolds in $\widehat{V}$ correspond to cylindrical (i.e. invariant with respect to the $\mathbb{R}_+$-action) Lagrangian submanifolds of $\widehat{V}$.

Projectivization

From the point of view desribed above, Contact geometry can be viewed as *projectivized* Symplectic geometry. Suppose the multiplicative group $\mathbb{R}_+$ or $\mathbb{R}^* = \mathbb{R} \setminus 0$, denoted G, acts on a $2n$-dimensional symplectic manifold (W, ω) by conformally symplectic transformation, i.e. $\lambda^*\omega = \lambda\omega$, $\lambda \in G$. If the quotient space $V = W/G$ is Hausdorff then it is automatically a $(2n-1)$-dimensional contact manifold. The contact plane $\xi_v \in T_vV$, $v \in V$, can be defined as follows. Take any point w from the orbit $v \subset W$ and consider the space $N_w = L_w^{\perp_\omega}$ where $L_w \subset T_wW$ is the tangent line to the orbit v at the point $w \in v$. Then ξ_v is the image $d\pi(N_v)$, where $\pi : W \to V = W/G$ is the canonical projection. Note that the standard contact structure on the space $V = PT^*M$ of contact elements or on the space $V = ST^*M$ of co-oriented contact elements is obtained by the above construction from $W = (T^*M \setminus M,\, d(p\,dq))$ using, respectively, the canonical action of $\mathbb{R}^*$ or $\mathbb{R}_+$.

I. Embeddings and immersions into contact manifolds

An immersion $f : V \to (W, \xi)$ is called *isotropic* if its differential df maps TV to $\xi \subset TW$. An isotropic immersion is called *Legendrian* (resp. *subcritical*) if $\dim V = n$ (resp. $\dim V < n$). It is important to observe that the differential df of an isotropic immersion f maps tangent spaces to V to subspaces of ξ which are *isotropic* with respect to the conformal symplectic structure $\mathrm{CS}(\xi)$. Indeed, we have $f^*(d\alpha) = df^*(\alpha) = 0$, where $\xi = \operatorname{Ker}\alpha$. In particular, the differential relation $\mathcal{R}_{\mathrm{Leg}} \subset J^1(V, W)$ responsible for Legendrian immersions consists of monomorphisms (fiberwise injective homomorphisms) $T_vV \to \xi$ onto Lagrangian subspaces of ξ.

For a contact manifold (W, ξ) a map $f : V \to (W, \xi)$ is called *contact* if it induces a contact structure on V. Such a map is automatically an immersion. It is important to observe that the intersection $df(TV) \cap \xi$ consist of symplectic subspaces with respect to the conformal symplectic structure $\mathrm{CS}(\xi)$. A monomorphism $F : TV \to TW$ is called *contact* if it is transversal to ξ (i.e. $F^{-1}(\xi)$ is a codimension one distribution on V) and the intersection $F(TV) \cap \xi$ consists of symplectic subspaces of ξ with respect to the conformal symplectic structure $\mathrm{CS}(\xi)$.

If the manifold V itself has a contact structure then we may consider *isometric contact* or *isocontact* maps $f : (V, \xi_V) \to (W, \xi_W)$ which induce on V the given structure ξ_V. If ξ_W and ξ_V are given by contact forms α_W and α_V then equivalently one can say that f is isocontact if $f^*\alpha_W = \varphi\alpha_V$ where $\varphi : V \to \mathbb{R}$ is a non-vanishing function. A monomorphism $F : TV \to TW$ is called *isocontact* if $\xi_V = F^{-1}(\xi_W)$ and F induces a conformally symplectic map $\xi_V \to \xi_W$ with respect to conformal symplectic structures $\mathrm{CS}(\xi_V)$ and $\mathrm{CS}(\xi_W)$.

◀ **Exercise.** Which of the differential relations $\mathcal{R}_{\mathrm{Leg}}$, $\mathcal{R}_{\mathrm{cont}}$, $\mathcal{R}_{\mathrm{isocont}}$ are open? Which are invariant with respect to $\operatorname{Diff} V$? ▶

9.5. Contact stability

By contact stability we mean the absence of non-trivial local invariants for objects related to contact structures. Here the locality refers both to the manifold itself and to the space of contact structures.

One can prove for contact structures stability results similar to Proposition 9.3.2 in the symplectic case.

Suppose we have a homotopy α_t, $t \in [0,1]$, of differential 1-forms on W; can we *conformally* realize this homotopy by an isotopy $\varphi_t : W \to W$ such that $\varphi_t^* \alpha_0 = e^{f_t}\alpha_t$? It is sufficient to find a corresponding time-dependent vector field v_t such that $\varphi_t = e^{tv_t}$. Differentiation of the equation $\varphi_t^*\alpha_0 = e^{f_t}\omega_t$ with respect to t give us the equation

$$\mathcal{L}_{v_t}\alpha_t = \dot{\alpha}_t + h_t\alpha_t \text{ for all } t \in [0,1]$$

with respect to v_t and a family of functions h_t.

9.5.1. (Solution to the equation $\mathcal{L}_{v_t}\alpha_t = \dot{\alpha}_t + h_t\alpha_t$ for a homotopy of contact forms) *Let α_t be a family of contact forms on a manifold W and ξ_t the family of contact structures which these forms define. Let v_t be the vector field on V which is characterized by the conditions*

- $\alpha_t(v_t) = 0$ *and*
- $v_t = I^{-1}_{(d\alpha_t)|_{\xi_t}}(\dot{\alpha}_t|_{\xi_t})$

Then $\mathcal{L}_{v_t}\alpha_t = \dot{\alpha}_t + h_t\alpha_t$ for a function $h_t : W \to \mathbb{R}$.

Proof. By Cartan's formula we have

$$\mathcal{L}_{v_t}\alpha_t = v_t \lrcorner\, d\alpha_t + d(\alpha_t(v_t)) = v_t \lrcorner\, d\alpha_t\,.$$

But

$$(v_t \lrcorner\, d\alpha_t)|_{\xi_t} = v_t \lrcorner\,(d\alpha_t|_{\xi_t}) = \dot{\alpha}_t|_{\xi_t}\,,$$

and hence

$$v_t \lrcorner\, d\alpha_t = \dot{\alpha}_t + h_t\alpha_t\,.$$

□

◀ **Remark.** The equation $\mathcal{L}_{v_t}\alpha_t = \dot{\alpha}_t$, which defines an isotopy which preserves the *form* α, can be solved near any submanifold $L \subset V$ provided the Reeb vector field R_α is transversal to L. ▶

The following theorem can be deduced from Lemma 9.5.1 in the same way as Theorem 9.3.2 was deduced from 9.3.1.

9.5.2. (Stability theorems)

(Stability near a compact subset) *Let ξ_t, $t \in [0,1]$, be a family of contact forms given in a neighborhood $\mathcal{O}p\, A \subset W$ of a compact subset $A \subset W$. Then there exists an isotopy $\varphi_t : \mathcal{O}p\, A \to W$ which is fixed on A such that $\varphi_t^*\xi_0 = \xi_t$, $t \in [0,1]$.*

(Darboux' theorem) *Locally any contact structure on a* $(2n+1)$*-dimensional manifold is equivalent to the standard contact structure* ξ_0 *on* $\mathbb{R}^{2n+1}$*. Moreover, locally any contact form on a* $(2n+1)$*-dimensional manifold is equivalent to the standard contact form* $\alpha_0 = \{dz - \sum_1^n y_i dx_i\}$ *on* $\mathbb{R}^{2n+1}$*.*

(Gray's theorem) *Let* ξ_t *,* $t \in [0,1]$*, be a family of contact structures on a closed manifold* V*. Then there exists an isotopy* $\varphi_t : V \to V$ *such that* $\varphi_t^*(\xi_0) = \xi_t$ *for* $t \in [0,1]$*.*

(Relative Gray's theorem) *Let* ξ_t *,* $t \in [0,1]$*, be a family of contact structures on a compact manifold* V *with boundary such that* $\xi_t \equiv \xi_0$ *on* $\mathcal{O}p\,V$*. Then there exists an isotopy* $\varphi_t : V \to V$ *fixed on* $\mathcal{O}p\,V$ *such that* $\varphi_t^*(\xi_0) = \xi_t$ *for* $t \in [0,1]$*.*

(Contact Weinstein's theorem) *Any isotropic immersion* $L \to W$*, and in particular any Legendrian one, extends to an isocontact immersion* $\mathcal{O}p\,L \to W$*, where* $\mathcal{O}p\,L$ *is a neighborhood of the zero-section* L *in the* 1*-jet space* $J^1(L)$ *endowed with its canonical contact structure.*

(Contact neighborhood theorem) *Let* $f : (V, \xi) \to (W, \xi')$ *be an isocontact immersion. Let* $E \to V$ *be the symplectic vector bundle whose fiber over a point* $v \in V$ *is the space* $\mathrm{CS}(\xi'_{f(v)})$*-dual to* $df(\xi_v) \subset \xi'_{f(v)}$*. Then* f *extends to an isocontact immersion* $\widehat{f} : (\mathcal{O}p\,V, \widetilde{\xi}) \to (W, \xi')$ *where* $\mathcal{O}p\,V$ *is a neighborhood of the* 0*-section* V *in the total space of the symplectic vector bundle* $E \to V$*, and* $\widetilde{\xi}$ *is the contact structure on* $\mathcal{O}p\,V \subset E$ *constructed in Lemma 9.4.1 above.*

◀ **Remark.** All the above statements hold parametrically. See, however, the remark after Theorem 9.3.2. ▶

9.6. Lagrangian and Legendrian submanifolds

Lagrangian submanifolds of symplectic manifolds and *Legendrian submanifolds* of contact manifolds play a central role in Symplectic topology.

A section $s : M \to T^*M$ is a Lagrangian embedding if and only if s is a closed 1-form. In fact, any Lagrangian submanifold L of T^*M can be viewed as a *multi-valued* closed 1-form. Any Lagrangian submanifold $L \subset T^*M$ can equivalently be characterized by the condition that the restriction $p\,dq|_L$ is a closed 1-form on L. A Lagrangian submanifold $L \subset T^*M$ is called *exact* if the closed 1-form $p\,dq|_L$ is exact. A Lagrangian *immersion* $f : L \to T^*M$ is called *exact* if the closed form $f^*(p\,dq)$ is exact.

A section $s : M \to J^1(M, \mathbb{R})$ is a Legendrian embedding if and only if s is the 1-jet extension J^1_f of a function $f : M \to \mathbb{R}$. In other words, Legendrian sections coincide with the holonomic sections of $J^1(M, \mathbb{R})$. Similarly to the case of Lagrangian submanifolds of the cotangent bundle, a general Legendrian submanifold of $J^1(M)$ corresponds to a graph ("wave front") of a *multivalued* function. The projection $J^1(M) = T^*(M) \times \mathbb{R} \to T^*M$ sends Legendrian submanifolds of $J^1(M, \mathbb{R})$ onto (immersed) exact Lagrangian submanifolds of T^*M. Conversely, any exact Lagrangian submanifold of T^*M lifts, uniquely up to a translation along the $\mathbb{R}$-factor, to a Legendrian submanifold of $J^1(M, \mathbb{R})$.

More generally, let (W, ω) be a symplectic manifold with an *exact* symplectic form $\omega = d\alpha$. A choice of a primitive α is sometimes referred to as a *Liouville structure* on W. Let us associate with a Liouville manifold (W, α) a contact manifold $\left(\widehat{W} = W \times \mathbb{R}\,,\, \xi = \operatorname{Ker}(dz - \alpha)\right)$, where z denotes the coordinate along the second factor. Let $\pi : \widehat{W} \to W$ be the projection to the first factor. A Lagrangian immersion $f : L \to (W, d\alpha)$ is called *exact* if the form $f^*\alpha$ is exact. If $H_1(L; \mathbb{R}) = 0$, then any Lagrangian immersion $L \to (W, d\alpha)$ is exact. The projection $f = \pi \circ \widehat{f}$ of a Legendrian immersion $\widehat{f} : L \to (\widehat{W}, \xi)$ into W is an exact Lagrangian immersion. Conversely, any exact Lagrangian immersion $f : L \to W$ lifts uniquely up to translation along the $\mathbb{R}$-factor to a Legendrian immersion $\widehat{f} : L \to \widehat{W}$ by the formula $\widehat{f} = (f, H)$, where $dH = f^*\alpha$. The relation between exact Lagrangian immersions into W and Legendrian immersions into $\widetilde{W}$ will be exploited later in Chapter 16.

We will show below (see 14.1 and 16.1) that isotropic and, in particular, Legendrian immersions into a contact manifold satisfy all forms of the h-principle. On the other hand, the literally understood h-principle fails for isotropic immersions into a symplectic manifold. Indeed, in order that a map $f : L \to (W, \omega)$ be homotopic to an isotropic immersion we have a necessary cohomological condition $f^*[\omega] = 0 \in H^2(L)$. Similarly, given a map $f : (L, A) \to W$ which is an isotropic immersion on a neighborhood $\mathcal{O}p\, A$ of a polyhedron $A \subset L$, vanishing of the relative cohomology class $[f^*\omega] \in H^2(L, A)$ is necessary in order that the map f be homotopic rel A to an isotropic immersion. As we will see in 14.1 and 16.3 below, the isotropic and, in particular, Lagrangian immersions into symplectic manifolds satisfy a modified form of the h-principle augmented by these additional necessary conditions.

Subcritical isotropic *embeddings* into contact and symplectic manifolds also satisfy some forms of the h-principle (see 12.4 below). However, the problems of *Lagrangian embeddings* and *Legendrian isotopy* belong to the world of symplectic and contact rigidity.

9.7. Hamiltonian and contact vector fields

A. Hamiltonian vector fields

A vector field X on a symplectic manifold (M,ω) is called *symplectic* if the Lie derivative $\mathcal{L}_X\omega$ vanishes, which is equivalent to the equation $d(X \lrcorner\, \omega) = 0$. If the form $X \lrcorner\, \omega$ is *exact*, i.e. $X \lrcorner\, \omega = dH$, then the vector field $X = X_H$ is called *Hamiltonian* and the function H is called the *Hamiltonian function* for the vector field X. If M is non-compact then we will always assume that X and H have compact supports. This condition defines H uniquely. On the other hand, if M is closed then H is defined up to an additive constant.

A time-dependent Hamiltonian function $H_t : M \to \mathbb{R}$, $t \in [0,1]$, defines a symplectic isotopy $\varphi_t = e^{tX_{H_t}} : M \to M,\ t \in [0,1]$. The isotopy φ_t is called a *Hamiltonian isotopy* with the time-dependent Hamiltonian function $H_t,\ t \in [0,1]$. A symplectomorphism $\varphi : M \to M$ is called *Hamiltonian* if it is the time 1 map of a Hamiltonian isotopy. The group $\mathrm{Ham} = \mathrm{Ham}\,(M,\omega)$ of Hamiltonian diffeomorphisms of (M,ω) is a normal subgroup of the identity component $\mathrm{Diff}_\omega = \mathrm{Diff}_\omega(M,\omega,\mathrm{Id}_M)$ of the group of symplectomorphisms of (M,ω). According to a theorem of A. Banyaga (see [**Ba78**]) $\mathrm{Ham} = [\mathrm{Diff}_\omega, \mathrm{Diff}_\omega]$.

It will be important for our applications to observe the following simple fact

9.7.1. (Symplectic cutting-off) *Let X be a Hamiltonian vector field on a symplectic manifold V. Then for any compact subset $A \subset V$ and its neighborhood $U \supset A$ there exists a Hamiltonian vector field $\tilde{X}_{A,U}$ which is supported in U and which coincides with X on A.*

Indeed, any Hamiltonian functions can be cut-off to 0 away from any given neighborhood $U \subset A$. $\square$

B. Contact vector fields

A vector field X on a contact manifold (V,ξ) is called *contact* if its flow (which is always defined at least locally) preserves the contact structure ξ. If $\xi = \mathrm{Ker}\,\alpha$ then this is equivalent to the equation $\mathcal{L}_X\alpha = h\alpha$ for a function $h : V \to \mathbb{R}$. Any contact vector field X on V lifts to a Hamiltonian vector field $\widehat{X}$ on the symplectization $\widehat{V}$ of V. The Hamiltonian $\widehat{K}_X$ which defines the field $\widehat{X}$ equals $\widehat{\alpha}_\xi(\widehat{X})$ where $\widehat{\alpha}_\xi$ is the canonical 1-form on the symplectization $\widehat{V}$. Note that $\widehat{\alpha}_\xi(\widehat{X})$ is a function on $\widehat{V}$ homogeneous of degree one (with respect to the canonical $\mathbb{R}_+$-action on $\widehat{V}$). Conversely, any

Hamiltonian $H : \widehat{V} \to \mathbb{R}$ that is homogeneous of degree one defines a vector field which projects to a contact vector field on V. A choice of a contact form α for ξ defines an embedding $f_\alpha : V \to \widehat{V}$ and a decomposition

$$X = a(v)R_\alpha + Y_\alpha\,,$$

where R_α is the Reeb vector field and $Y_\alpha \in \xi$. The function $K_{X,\alpha} : V \to \mathbb{R}$ induced by the embedding f_α from the Hamiltonian function $\widehat{K}_X$ is equal to $\alpha(X) = a(v)$. It is called the *contact Hamiltonian* for the contact vector field X (with respect to α). Thus the contact Hamiltonian measures the component transversal to ξ of the field X. It follows that the transversal component of X completely determines X. In particular, *there are no non-zero contact vector fields which are everywhere tangent to ξ.* Note that the Reeb vector field R_α is itself a contact vector field which is defined by the contact Hamiltonian $K \equiv 1$. Any contact vector field X which is transversal to ξ is the Reeb vector field for the contact form α, $\operatorname{Ker}\alpha = \xi$, characterized by the equation $\alpha(X) = 1$.

Similarly to the symplectic case we have the following

9.7.2. (Contact cutting-off) *Let X be a contact vector field on a contact manifold V. Then for any compact subset $A \subset V$ and its neighborhood $U \supset A$ there exists a contact vector field $\tilde{X}_{A,U}$ which is supported in U and which coincides with X on A.*

Indeed, any contact Hamiltonian functions can be cut-off to 0 away from any given neighborhood $U \subset A$.

Chapter 10

Symplectic and Contact Structures on Open Manifolds

10.1. Classification problem for symplectic and contact structures

A. Symplectic structures

A symplectic structure ω on V defines a volume form ω^n and hence an orientation of V. Thus we should consider only even-dimensional *orientable* manifolds. Given such a manifold V we will use the following notation:

- $\mathcal{J} = \mathcal{J}(TV)$ - the space of almost complex structures on V = the space of complex structures on the tangent bundle TV;
- $\mathcal{S}_{\mathrm{symp}} = \mathcal{S}(TV)$ - the space of almost symplectic structures on V = the space of symplectic structures on the tangent bundle TV;
- $\mathbb{S}_{\mathrm{symp}}$ - the space of symplectic structures on V;
- $\mathbb{S}^a_{\mathrm{symp}}$ - the space of symplectic structures on V in a given cohomology class $a \in H^2(V, \mathbb{R})$.

The existence of an almost symplectic (or, equally, an almost complex) structure on V is a necessary condition for the existence of a symplectic structure. The existence of a homotopy in $\mathcal{S}_{\mathrm{symp}}$ which connects two given symplectic forms ω_0 and ω_1 is a necessary condition for the existence of a symplectic homotopy between ω_0 and ω_1, and so on. Thus according to the philosophy of the h-principle the problem of *homotopical* classification of symplectic structures on a manifold V can be treated as the study of homotopy properties of the natural inclusions

$$\mathbb{S}_{\text{symp}} \hookrightarrow \mathcal{S}_{\text{symp}} \ \text{ or } \ \mathbb{S}^a_{\text{symp}} \hookrightarrow \mathcal{S}_{\text{symp}}\,.$$

◀ **Remark.** Let us recall that $\mathcal{J}$ is homotopy equivalent to $\mathcal{S}_{\text{symp}}$, and hence, considering the classification of symplectic structures up to homotopy, one can equally use $\mathcal{S}_{\text{symp}}$ or $\mathcal{J}$ as the corresponding "formal" space. See also the Remark in 9.2 **A**. ▶

B. Contact structures

An *almost contact* structure on a $(2n+1)$-dimensional manifold V is a codimension one tangent distribution ξ on V together with a symplectic form ω on ξ valued in the 1-dimensional bundle $\lambda = TV/\xi$. Note that if n is odd then an almost contact structure defines an orientation of V; if n is even it defines an orientation of ξ. Note that if ξ is *cooriented* then from the homotopical point of view defining a form ω valued in λ is the same as defining a *positive-conformal* class of a symplectic structure on ξ. Hence we can define a *cooriented* almost contact structure on a $(2n+1)$-dimensional manifold V as a pair (ξ_+, ω) where ξ_+ a cooriented hyperplane distribution on V and ω a positive-conformal class of symplectic structures on ξ_+.

Given an odd-dimensional manifold V, we will use the following notation:

- $\mathcal{S}_{\text{cont}}$ - the space of almost contact structures on V;
- $\mathbb{S}_{\text{cont}}$ - the space of contact structures on V;
- $\mathcal{S}^+_{\text{cont}}$ - the spaces of cooriented almost contact structures on V;
- $\mathbb{S}^+_{\text{cont}}$ - the spaces of cooriented contact structures on V.

According to the philosophy of the h-principle the problem of *homotopical* classification of contact structures on a manifold V can be treated as the study of the homotopy properties of the natural inclusions

$$\mathbb{S}_{\text{cont}} \hookrightarrow \mathcal{S}_{\text{cont}} \ \text{ or } \ \mathbb{S}^+_{\text{cont}} \hookrightarrow \mathcal{S}^+_{\text{cont}}\,.$$

10.2. Symplectic structures on open manifolds

Let us recall that $\Lambda^p V$ is a natural vector bundle: any diffeomorphism $h : V \to V$ lifts to $\Lambda^p V$ as the exterior power $d^p h$ of the differential $dh : TV \to TV$. Hence we can consider $\operatorname{Diff} V$-invariant subspaces of $\Lambda^p V$.

For a subspace $\mathcal{R} \subset \Lambda^p V$ and a cohomology class $a \in H^p(V)$ we denote by $\mathrm{Clo}_a \mathcal{R}$ a subspace of the space $\operatorname{Sec} \mathcal{R}$ which consists of *closed* p-forms $\omega : V \to \mathcal{R}$ in the cohomology class a.

10.2.1. *Let V be an open manifold, $a \in H^p(V)$ a fixed cohomology class and $\mathcal{R} \subset \Lambda^p V$ an open* Diff V*-invariant subset. Then the inclusion*

$$\mathrm{Clo}_a \mathcal{R} \hookrightarrow \mathrm{Sec}\, \mathcal{R}$$

is a homotopy equivalence. In particular,

- *any p-form $\omega : V \to \mathcal{R}$ is homotopic in $\mathcal{R}$ to a closed p-form $\overline{\omega} \in a$;*
- *any homotopy of p-forms $\omega_t : V \to \mathcal{R}$ which connects two closed forms $\omega_0, \omega_1 \in a$ can be deformed in $\mathcal{R}$ into a homotopy of closed forms $\overline{\omega}_t \in a$ connecting ω_0 and $\omega_1 \in a$.*

Proof. The statement follows almost immediately from Theorem 4.7.4. We explain the reduction in the non-parametric case, the general case differs only in notation. Let $K \subset V$ be a polyhedron of positive codimension, as in 4.3.1. According to Theorem 4.7.2 there exist a diffeotopy $h^\tau : V \to V$ and a closed form $\widetilde{\omega} \in a$ which is arbitrarily C^0-close to ω over a neighborhood U of $\widetilde{K} = h^1(K) \subset V$. Hence over the neighborhood U the linear homotopy ω_t between ω and $\widetilde{\omega}$ lies in $\mathcal{R}$. Let $g_t : V \to V$ be a diffeotopy which compresses V into U. Then $\overline{\omega} = (g_1^{-1})^* \widetilde{\omega}$ is a section of $\mathcal{R}$ and $\overline{\omega} \in a$. Applying consecutively the homotopies $(g_t^{-1})^* \omega$ and $(g_1^{-1})^* \omega_t$ we get the required homotopy which connects ω and $\overline{\omega}$ in $\mathcal{R}$. ☐

For a $2n$-dimensional manifold V let $\mathcal{R}_{\mathrm{symp}} \subset \Lambda^2 V$ be defined in every fiber by the condition $\beta^n \neq 0$. Then $\mathrm{Clo}_a \mathcal{R}_{\mathrm{symp}} = \mathbb{S}^a_{\mathrm{symp}}$ and $\mathrm{Sec}\, \mathcal{R}_{\mathrm{symp}} = \mathcal{S}_{\mathrm{symp}}$. The set $\mathcal{R}_{\mathrm{symp}}$ is open and Diff V-invariant. Therefore, applying 10.2.1 we get the following homotopy principle for symplectic forms on open manifolds:

10.2.2. (Gromov [Gr69]) *For any open manifold V the inclusion*

$$\mathbb{S}^a_{\mathrm{symp}} \hookrightarrow \mathcal{S}_{\mathrm{symp}}$$

is a homotopy equivalence. In particular,

- *any 2-form $\beta \in \mathcal{S}_{\mathrm{symp}}$ is homotopic in $\mathcal{S}_{\mathrm{symp}}$ to a symplectic form ω, $\omega \in a \in H^2(V)$;*
- *if two symplectic forms $\omega_0, \omega_1 \in \mathbb{S}^a_{\mathrm{symp}}$ are homotopic in $\mathcal{S}_{\mathrm{symp}}$ then ω_0 and ω_1 are homotopic in $\mathbb{S}^a_{\mathrm{symp}}$.*

Using almost complex structures (instead of almost symplectic) we can formulate the existence theorem in the following way: any open almost complex manifold (M, J) admits a symplectic structure ω which belongs to any prescribed cohomology class $a \in H^2(M)$ and such that $J \in [J_\omega]$ where $[J_\omega]$ is the homotopy class of almost complex structures compatible with ω. It is important to understand that we do not assert that $J \in J_\omega$, i.e. ω may be non-compatible with the *original* J.

◀ **Exercise.** Prove that the inclusion $\mathbb{S}_{\text{symp}} \hookrightarrow \mathcal{S}_{\text{symp}}$ is a homotopy equivalence. ▶

10.3. Contact structures on open manifolds

For a subset $\mathcal{R} \subset \Lambda^{p-1} \oplus \Lambda^p V$ let $\operatorname{Exa}\mathcal{R} \subset \operatorname{Sec}\mathcal{R}$ be the subspace of pairs

$$(\alpha, \beta) : V \to \Lambda^{p-1}V \oplus \Lambda^p V$$

such that $\beta = d\alpha$.

10.3.1. *Let V be an open manifold and $\mathcal{R}$ an open $\operatorname{Diff} V$-invariant subset of $\Lambda^{p-1} \oplus \Lambda^p V$. Then the inclusion*

$$\operatorname{Exa}\mathcal{R} \hookrightarrow \operatorname{Sec}\mathcal{R}$$

is a homotopy equivalence. In particular, any section $(\alpha, \beta) : V \to \mathcal{R}$ is homotopic in $\mathcal{R}$ to a section $(\bar{\alpha}, d\bar{\alpha}) : V \to \operatorname{Exa}\mathcal{R}$.

Proof. To simplify notation we will discuss only the non-parametric case. Let $K \subset V$ be a polyhedron of positive codimension, as in 4.3.1. According to Theorem 4.7.1 there exist a diffeotopy $h^\tau : V \to V$ and a $(p-1)$-form $\widetilde{\alpha}$ such that the pair $(\widetilde{\alpha}, d\widetilde{\alpha})$ is arbitrarily C^0-close to (α, β) over a neighborhood U of $\widetilde{K} = h^1(K) \subset V$. Hence over U the linear homotopy (α_t, β_t) between (α, β) and $(\widetilde{\alpha}, d\widetilde{\alpha})$ lies in $\mathcal{R}$. Let $g_t : V \to V$ be a diffeotopy which compresses V into U. Then $(\bar{\alpha}, d\bar{\alpha}) = (g_1^{-1})^*(\widetilde{\alpha}, d\widetilde{\alpha})$ is a section of $\mathcal{R}$. Consecutively applying the homotopies $(g_t^{-1})^*(\alpha, \beta)$ and $(g_1^{-1})^*(\alpha_t, \beta_t)$ we construct the required homotopy which connects (α, β) and $(\bar{\alpha}, d\bar{\alpha})$ in $\mathcal{R}$. □

Now let $\dim V = 2n+1$ and the set $\mathcal{R}_{\text{cont}} \subset \Lambda^1 V \oplus \Lambda^2 V$ be defined in every fiber by the condition $\alpha \wedge (\beta^n) \neq 0$. Then we have the following commutative diagram

$$\begin{array}{ccc} \operatorname{Exa}\mathcal{R}_{\text{cont}} & \hookrightarrow & \operatorname{Sec}\mathcal{R}_{\text{cont}} \\ \downarrow & & \downarrow \\ \mathbb{S}^+_{\text{cont}} & \hookrightarrow & \mathcal{S}^+_{\text{cont}} \end{array}$$

where the vertical arrows are natural homotopy equivalences. The set $\mathcal{R}_{\text{cont}}$ is open and $\operatorname{Diff} V$-invariant. Hence, Theorem 10.3.1 implies the following homotopy principle for cooriented contact structures on open manifolds.

10.3.2. (Gromov [Gr69]) *For any open manifold V the embedding*

$$\mathbb{S}^+_{\text{cont}} \hookrightarrow \mathcal{S}^+_{\text{cont}}$$

is a homotopy equivalence.

In particular, *given an open manifold M, a non-vanishing 1-form α_0 on M and an almost complex structure on the bundle $\xi_0 = \operatorname{Ker} \alpha_0$, there exist a family of non-vanishing 1-forms α_t on M and a family of almost complex structures J_t on $\xi_t = \operatorname{Ker} \alpha_t$, $t \in [0,1]$, such that α_1 is a contact form and J_1 is compatible with the symplectic form $d\alpha_1|_{\xi_1}$.* Note that even in the case of an orientable 3-dimensional manifold M the latter statement implies more than just the existence of a contact structure in every homotopy class of a coorientable tangent plane field. It asserts in addition that such a structure can be chosen to define an a priori given orientation of the manifold M.

Theorem 10.3.2 can be generalized to cover the case of not necessarily coorientable contact structures. As we explained in Section 9.4C above, any tangent hyperplane field ξ on M can be defined by a Pfaffian equation $\alpha = 0$, where the 1-form α is valued in the not necessarily trivial line bundle $\Lambda = TL/\xi$. The contact condition for ξ means, as usual, that $d\alpha|_\xi$ is non-degenerate, where the form $d\alpha$ is also valued in L. It is straightforward to extend Theorem 10.3.1 to cover the case of relations $\mathcal{R} \subset (\Lambda^{p-1} \otimes L) \oplus (\Lambda^p V \otimes L)$, which then implies the corresponding generalization of Theorem 10.3.2 to the general case of not necessarily coorientable contact structures.

Both h-principles 10.2.2 and 10.3.2 also hold in the relative version, when one wants to extend a symplectic or contact structure from a neighborhood of a subcomplex of codimension > 1.

10.4. Two-forms of maximal rank on odd-dimensional manifolds

The rank of a differential 2-form is always even, and hence the maximal rank of a 2-form on a manifold of dimension $2n+1$ equals $2n$. Theorem 10.2.2 implies the h-principle for such forms on *any* (open or closed) manifold. It was first proved by D. McDuff using the convex integration technique, see Section 20.5.

Given an odd-dimensional manifold V, we will use the following notation

- $\mathcal{S}_{\mathrm{non-deg}}$ - the space of 2-forms on V of maximal rank;
- $\mathbb{S}^a_{\mathrm{non-deg}}$ - the subspace in $\mathcal{S}_{\mathrm{non-deg}}$ which consists of closed forms in a given cohomology class $a \in H^2(V)$.

10.4.1. (McDuff [MD87a]) *The inclusion*

$$\mathbb{S}^a_{\mathrm{non-deg}}(V) \hookrightarrow \mathcal{S}_{\mathrm{non-deg}}(V)$$

is a homotopy equivalence. In particular, if V admits a 2-form of maximal rank then every two-dimensional cohomology class of V can be represented by a closed non-degenerate form.

Proof. If V is orientable then any non-degenerate 2-form on V extends in a homotopically unique way to a non-degenerate 2-form on $V \times \mathbb{R}$. Conversely, the restriction of a symplectic form on $V \times \mathbb{R}$ to $V = V \times 0 \subset V \times \mathbb{R}$ is a non-degenerate closed 2-form. Hence, the required homotopy equivalence follows in this case from the h-principle for symplectic forms on open manifolds, see 10.2.2.

If V is non-orientable, then with any non-degenerate 2-form ω on V we associate its kernel $\operatorname{Ker}\omega$, a non-orientable line subbundle of TV. The form ω homotopically canonically extends as a non-degenerate form to the total space K of this line bundle. Hence we can apply Theorem 10.2.2 to produce a symplectic form $\widetilde{\omega}$ on K in a given cohomology class and then restrict it back to the 0-section. □

The same argument proves the relative version of the above h-principle. A contact analog of Theorem 10.4.1 will be discussed in Section 14.2 below.

Chapter 11

Symplectic and Contact Structures on Closed Manifolds

In general the problem of constructing symplectic or contact structures on closed manifolds, or the problem of extending the structures from a codimension one subpolyhedron do not abide by any h-principle. In this section we review the current knowledge about this subject.

11.1. Symplectic structures on closed manifolds

A. Homotopy and isotopy

Two symplectic forms ω_0 and ω_1 on a manifold V are called

- *homotopic* if they are homotopic in $\mathbb{S}_{\mathrm{symp}}$;
- *formally homotopic* if they are homotopic in $\mathcal{S}_{\mathrm{symp}}$;
- *isotopic* if there exist an isotopy $\varphi_t : V \to V$ such that $\varphi_1^*\omega_0 = \omega_1$.

Theorem 9.3.2 implies that classification of symplectic structures on a *closed* manifold V up to *homotopy* in a *fixed cohomology class* coincides with the classification up to *isotopy*:

$$\omega_0 \text{ and } \omega_1 \text{ are isotopic} \quad \Leftrightarrow \quad \text{they are homotopic in} \quad \mathbb{S}^a_{\mathrm{symp}}\,.$$

B. Existence and uniqueness

For a closed symplectic manifold (V, ω) the cohomology class $[\omega] \in H^2(V; \mathbb{R})$ represented by the closed form ω satisfies the inequality $[\omega]^n \neq 0$. Hence, if understood literally the h-principle for the inclusions

$$\mathbb{S}_{\text{symp}}(V) \hookrightarrow \mathcal{S}_{\text{symp}}(V) \text{ and } \mathbb{S}^a_{\text{symp}}(V) \hookrightarrow \mathcal{S}_{\text{symp}}(V)$$

may fail for an almost symplectic closed manifolds V of dimension > 2 because of the cohomological obstruction: a symplectic candidate $V = V^{2n}$ should at least have a cohomology class $a \in H^2(V;\mathbb{R})$ with $a^n \neq 0$. However, with this modification the situation becomes far less clear.

(a) *Does any closed manifold V^{2n} which admits an almost symplectic structure ω_0 and a cohomology class $a \in H^2(V)$ with $a^n \neq 0$ have a symplectic structure ω_1 such that $[\omega_1] = a \in H^2(V)$ and ω_1 is formally homotopic to ω_0? Does V admit any symplectic structure at all?*

(b) *Let ω_0, ω_1 be two symplectic structures on V such that ω_0 is formally homotopic to ω_1 and the cohomology classes $[\omega_0]$ and $[\omega_1]$ coincide. Are ω_0 and ω_1 isotopic?*

(c) *Let ω_0, ω_1 be two symplectic structures on V such that ω_0 is formally homotopic to ω_1. Are ω_0 and ω_1 homotopic?*

◀ **Exercise.** Formulate the questions a) - c) in terms of homotopy properties of the inclusions $\mathbb{S}_{\text{symp}}(V) \hookrightarrow \mathcal{S}_{\text{symp}}(V)$ and $\mathbb{S}^a_{\text{symp}}(V) \hookrightarrow \mathcal{S}_{\text{symp}}(V)$ ▶

The answer to a) is completely open for manifolds of dimension $2n > 4$. Thus it is still possible, though unlikely, that the problem a) abides by the h-principle. For $n = 2$ the answer is known to be negative. For instance, C.H. Taubes (see [**Ta94**]) proved that the connected sum of odd numbers of copies of $\mathbb{C}P^2$ has no symplectic structure, while it admits an almost symplectic structure and also satisfies the cohomological condition.

D. McDuff constructed in [**MD87b**] an example of a 6-manifold which has two non-isotopic but homotopic symplectic forms in the same cohomology class. In particular, this implies the negative answer to b) for manifolds of dimension > 4. In dimension 4 the problem b) is still open. Note that C. McMullen and C. Taubes (see [**MT99**]) constructed an example of a simply-connected 4-manifold which admits symplectic forms whose first Chern classes are not equivalent under the action of the diffeomorphism group.

The answer to c) is negative for manifolds of dimension > 4. A counterexample was constructed by Y. Ruan, see [**Ru94**], using Gromov's theory of holomorphic curves in symplectic manifolds.

C. Extension problems

Let ω be a symplectic form on $\mathcal{O}p\,\partial D^{2n}$. If $n = 1$ it obviously extends to D^2. If $n > 1$ then besides the homotopical obstruction, i.e. existence of a (not necessarily closed) non-degenerate form extending ω, there is another obstruction which is similar to the cohomological obstruction for closed manifolds which we discussed above. Namely, any extension $\widetilde{\omega}$ of ω to D^{2n} is exact, $\widetilde{\omega} = d\widetilde{\theta}$, and by Stokes' theorem we have

$$\int\limits_{D^{2n}} \widetilde{\omega}^n = \int\limits_{\partial D^{2n}} \widetilde{\theta} \wedge \omega^{n-1} .$$

The first integral is positive. Hence, the second integral is positive as well. But this integral is independent of the choice of a primitive of the form $\omega|_{\partial D^{2n}}$, and hence it does not depend on the choice of $\widetilde{\omega}$. We will refer to the positivity of this integral as the *positivity condition.*

(a) Let ω be a symplectic form on $\mathcal{O}p\,\partial D^{2n}$, $n > 1$, which extends to D^{2n} as a non-degenerate form and which satisfies the positivity condition. *Does it extend to D^{2n} as a symplectic form?*

(b) Let ω be a symplectic form on D^{2n} which coincides with the standard symplectic form $\omega_0 = \sum\limits_1^n dp_i \wedge dq_i$ on $\mathcal{O}p\,\partial D^{2n}$. *Is there a diffeomorphism $f : D^{2n} \to D^{2n}$ fixed on $\mathcal{O}p\,\partial D^{2n}$ such that $f^*\omega = \omega_0$? Can such f be chosen isotopic to the identity relative to the boundary?*

The answer to a) is negative in all dimensions. This can be deduced from Gromov's non-squeezing theorem, see [**Gr86**]. [1] Problem b) is open in dimension $2n > 4$. In the 4-dimensional case a theorem of Gromov (see [**Gr85**]) asserts that the answer is positive to the question about existence of a diffeomorphism f with $f^*\omega = \omega_0$. However it is unknown whether f can be chosen isotopic to the identity.

11.2. Contact structures on closed manifolds

A. Homotopy and isotopy

Two contact structures ξ_0 and ξ_1 on a manifold V are called

- *homotopic* if they are homotopic in $\mathbb{S}_{\mathrm{cont}}$;
- *formally homotopic* if they are homotopic in $\mathcal{S}_{\mathrm{cont}}$;
- *isotopic* if there exists an isotopy $\varphi_t : V \to V$ such that $\varphi_1^*\xi_0 = \xi_1$.

[1] As far as the authors know, the proof of this result is not published anywhere.

Theorem 9.5.2 implies that the classification of contact structures on a *closed* manifold V up to *homotopy* coincides with the classification up to isotopy:

$$\xi_0 \text{ and } \xi_1 \text{ are isotopic} \quad \Leftrightarrow \quad \text{they are homotopic}\,.$$

B. Homotopy principle

Question: *Does the h-principle 10.3.2 hold for closed manifolds V?*

The answer to this question is completely open for manifolds of dimension > 3. However, in the 3-dimensional case the situation is not that bad. In particular,

11.2.1. (J. Martinet [Ma71], R. Lutz [Lu77]) *Any plane field ξ on an orientable closed 3-manifold V is homotopic to a contact structure which defines an a priori given orientation of V. In other words, the h-principle 10.3.2 holds on the level of an epimorphism on π_0.*

However,

11.2.2. (D. Bennequin [Be83]) *There exists a contact structure ζ on S^3, which is homotopic to the standard contact structure ξ on S^3 as a plane field, defines the same orientation of S^3 as ξ, but which is not equivalent to ξ. In other words, the h-principle 10.3.2 fails on the level of a monomorphism on π_0.*

The contact structure ζ in the previous theorem is *overtwisted.* This means that there exists an embedded disc $D \subset V$ which is tangent to ξ along the boundary ∂D. A disc with this property is also called overtwisted. D. Bennequin proved in [**Be83**] that in the standard contact structure ξ on S^3 such an overtwisted disc cannot be found.

It turned out that overtwisted contact structures do abide by an *h*-principle.

11.2.3. (Y. Eliashberg [El89]) *Let V be an oriented manifold and*

$$\mathrm{Cont}_{\mathrm{ot}}(V, D) \subset \mathbb{S}_{\mathrm{cont}}$$

be the space of positive overtwisted contact structures on V which all coincide in a neighborhood of a disc $D \subset V$ and which have D as an overtwisted disc. Let

$$\mathrm{Distr}\,(V, D) \subset \mathcal{S}_{\mathrm{cont}}$$

be the space of tangent plane fields η on V, such that $\eta_p = T_p(D)$, where p is a fixed point of the disc D. Then the inclusion

$$\mathrm{Cont}_{\mathrm{ot}}(V, D) \hookrightarrow \mathrm{Distr}\,(V, D)$$

is a homotopy equivalence. In particular, if two overtwisted contact structures of the same orientation are homotopic as plane fields, then they are isotopic.

The dichotomy of contact structures on 3-manifolds into overtwisted and the complementary class of structures, called *tight*, turned out to be quite productive. In particular, tight contact structures have been classified on many closed 3-manifolds, including S^3, lens spaces, torus bundles over S^1 and circle bundles over surfaces, see [**El91**], [**Gi99**], [**Et99**], [**Gi01**], [**Ho00**]. This classification shows, in particular, that the h-principle for tight contact structures fails even on the level of an epimorphism on π_0.

C. Extension problem

Does the h-principle hold for the extension problem of contact structures from $\mathcal{O}p\,\partial D^{2n+1}$ to D^{2n+1}? In particular,

(a) Suppose a contact form α on $\mathcal{O}p\,\partial D^{2n+1}$ formally extends to D^{2n+1}, i.e. there exists a pair $\tilde{\alpha}, \omega$ on D^{2n+1} such that the non-vanishing form $\tilde{\alpha}$ extends α, the 2-form ω extends $d\alpha$, and $\omega|_{\tilde{\xi}=\operatorname{Ker}\tilde{\alpha}}$ is non-degenerate. *Does α extend to D^{2n+1} as a contact form?*

(b) Let α be a contact form on D^{2n+1} which coincides over $\mathcal{O}p\,\partial D^{2n+1}$ with the standard contact 1-form $\alpha_0 = dz - \sum_1^n p_i dq_i$. Suppose that α_0 and α are formally homotopic relative to the boundary, i.e. $(\alpha, d\alpha)$ and $(\alpha_0, d\alpha_0)$ are homotopic through sections of $\mathcal{R}_{\mathrm{cont}}$ which coincide with $(\alpha_0, d\alpha_0)$ over $\mathcal{O}p\,\partial D^{2n+1}$. *Is there a diffeotopy of D^{2n+1} fixed over $\mathcal{O}p\,\partial D^{2n+1}$ which moves α into α_0?*

The situation with these problems is very much the same as in the closed case.

In the 3-dimensional case the answer to a) is positive. It is also positive in the class of overtwisted contact structures. However, the answer is negative in the case of tight structures. According to Bennequin's theoren 11.2.2 the answer to b) is negative if the structure α is overtwisted. However if α is tight, then the answer to b) is positive, see [**El91**].

If $2n+1 > 3$ then much less is known. Problem a) is completely open in this case. Ustilovsky's theorem [**Us99**] shows that the answer to b) is negative if n is even. It is also negative for selected even n (for instance, for $n = 3$ see [**GT99**]). The problem is open otherwise.

Chapter 12

Embeddings into Symplectic and Contact Manifolds

12.1. Isosymplectic embeddings

Let us recall that when considering embeddings and immersions into a symplectic manifold (W, ω_W) we differentiate between immersions and embeddings which induce on the source a symplectic structure, and immersions and embeddings of another symplectic manifold which induce on it the symplectic structure which was a priori given. The mappings of the first kind we call symplectic, while the second kind mappings we call isometric symplectic, or isosymplectic.

◀ **Remark.** The term "isometric symplectic" was used by G. D'Ambra and A. Loi (see [**DL01**]) in a different sense. Namely for symplectic manifolds endowed with compatible almost complex structures, and therefore Riemannian metrics, they considered the problem of finding immersions which are simultaneously isosymplectic and isometric with respect to the given Riemannian metrics. They proved in [**DL01**] an h-principle type result which is a mixture of Gromov's Theorem 12.1.1 and Nash-Kuiper's C^1-isometric immersion theorem, see Theorem 21.2.1 below. ▶

The symplectic condition is open (while the isosymplectic is not!). Hence, for an open manifold V Theorem 7.2.3 implies the parametric h-principle for symplectic *immersions*, while Theorem 4.5.1 yields a theorem about directed symplectic *embeddings*. We turn now to the problem of isosymplectic embeddings and immersions.

Let (V, ω_V) and (W, ω_W) be symplectic manifolds. A monomorphism (fiberwise injective homomorphism) $F : TV \to TW$ which covers a map $f = \mathrm{bs}F : V \to W$ is called *symplectic* if $F^*\omega_W$ is non-degenerate and the equality $f^*[\omega_W] = [\omega_V]$ holds for the cohomology classes. A monomorphism $F : TV \to TW$ which covers a map $f = \mathrm{bs}F : V \to W$ is called *isosymplectic* if $F^*\omega_W = \omega_V$ and the equality $f^*[\omega_W] = [\omega_V]$ holds for the cohomology classes.

Let us recall (see Section 4.3) that for an open manifold V a polyhedron $V_0 \subset V$ is called a *core* of V if for an arbitrarily small neighborhood U of V_0 there exists an isotopy $h_t : V \to V$ *fixed on* V_0 which brings V to U.

12.1.1. (Isosymplectic embeddings, Gromov [Gr86]) *Let (V, ω_V) and (W, ω_W) be symplectic manifolds of dimension $n = 2m$ and $q = 2l$ respectively. Suppose that an embedding $f_0 : V \to W$ satisfies the cohomological condition $f^*[\omega_W] = [\omega_V]$, and the differential $F_0 = df_0$ is homotopic via a homotopy of monomorphisms*

$$F_t : TV \to TW\,, \quad \mathrm{bs}\, F_t = f_0$$

to an isosymplectic homomorphism $F_1 : TV \to TW$.

- **Open case.** *If $n \le q - 2$ and the manifold V is open then there exists an isotopy $f_t : V \to W$ such that the embedding $f_1 : V \to W$ is isosymplectic and the differential df_1 is homotopic to F_1 through isosymplectic homomorphisms. Moreover, given a core $V_0 \subset V$, one can choose the isotopy f_t to be arbitrarily C^0-close to f_0 near V_0.*
- **Closed case.** *If $n \le q-4$ then the above isotopy f_t exists even if V is closed. Moreover, one can choose the isotopy f_t to be arbitrarily C^0-close to f_0.*

Proof of 12.1.1 in the open case. The construction of f_t proceeds in two steps.

Step 1. The homotopy F_t gives us a homotopy $G_t = \mathrm{G}F_t$ between the tangent lift $G_0 = \mathrm{G}df_0$ of f_0 and a map

$$G_1 = \mathrm{G}F_1 : V \to A_{\mathrm{symp}} \subset \mathrm{Gr}_n W\,.$$

The set $A_{\mathrm{symp}} \subset \mathrm{Gr}_n W$ is open and hence, according to Theorem 4.5.1, there exists an isotopy $\widetilde{f}_t : V \to W$, $\widetilde{f}_0 = f_0$, such that $\widetilde{f}_t$ is C^0-small near the core V_0 and $\widetilde{f}_1 : V \to W$ is a *symplectic embedding* whose tangential lift $\mathrm{G}d\widetilde{f}_1$ is homotopic to G_1. According to 4.5.2 one can additionally arrange that the differential $d\widetilde{f}_1$ and the isosymplectic homomorphism F_1 are homotopic via a homotopy Φ_t for which $\mathrm{G}\Phi_t(V) \subset A_{\mathrm{symp}}$. This yields a homotopy of *non-degenerate* forms which connects $\widetilde{f}_1^*\omega_W$ and $F_1^*\omega_W = \omega_V$. This homotopy combined with the condition $[\omega_V] = [f_0^*\omega_W]$ allows us to apply Theorem 10.2.2 to get a homotopy ω_t of *symplectic* forms on V such that

$$\omega_0 = \widetilde{f}_1^*\omega_W\,, \quad \omega_1 = \omega_V \quad \text{and} \quad [\omega_t] = \text{const} \in H^2(V;\mathbb{R})\,.$$

Step 2. Note that $\omega_t = \omega_0 + d\alpha_t$ where α_t is a homotopy of 1-forms on V. Hence, Theorem 12.1.1 in the open case follows from

12.1.2. (Realization of an exact homotopy of symplectic forms by an isotopy of symplectic embeddings) *Let V be an arbitrary (open or closed) manifold of dimension $n = 2m$ and $h_0 : V \to (W, \omega_W)$ a symplectic embedding into a symplectic manifold (W, ω_W) of dimension $q = 2l > n$. Set $\omega_0 = h_0^*\omega_W$. Let $\omega_t = \omega_0 + d\alpha_t$, $t \in [0,1]$, be a homotopy of symplectic forms on V. Then there exists an arbitrarily C^0-small symplectic isotopy $h_t : V \to (W, \omega_W)$ such that $h_1^*\omega_W = \omega_1$.*

◄ **Remarks**

1. If the manifold V is closed then Moser's theorem 9.3.2 ensures existence of an isotopy $h_t : V \to V$ such that $h_t^*(\omega_0) = \omega_0 + d\alpha_t$, $t \in [0,1]$. However this isotopy cannot be chosen C^0-small. If V has a boundary then Moser's theorem is no longer true unless α_t vanishes on the boundary. Note that Proposition 12.2.2 below shows what can be salvaged from Moser's theorem in the case of manifolds with boundary.

2. The proof below can easily be adjusted so it would work *parametrically.* In particular, under the assumptions of Lemma 12.1.2 *there exists an isotopy $\widetilde{h}_t : V \to (W, \omega_W)$, with $\widetilde{h}_0 = h_0$ such that $\widetilde{h}_t^*\omega_W = \omega_0 + d\alpha_t$, $t \in [0,1]$.* ►

Proof of Lemma 12.1.2. A differential 1-form $r\,ds$ where r and s are compactly supported functions $V \to \mathbb{R}$ is called *primitive.* A homotopy of 1-forms β_t is called *piecewise primitive* if β_t linearly interpolates a (finite or infinite) sequence of 1-forms β_i, $i = 0, 1, \dots$, such that $\beta_{i+1} - \beta_i$ is a primitive 1-form.

In order to prove Proposition 12.1.2 we first approximate the homotopy α_t by a piecewise primitive homotopy $\widetilde{\alpha}_t$ such that $\omega_0 + d\widetilde{\alpha}_t$ still is a homotopy of *symplectic* forms and then realize each corresponding local homotopy of 2-forms

$$\omega_i + (t - t_i)\, dr_{i+1} \wedge ds_{i+1},\ t \in [t_i, t_{i+1}],$$

by an isotopy of embeddings. Here (and below) we assume that the approximating homotopy coincides with α_t for $t = 0, 1$.

Note that we can approximate the homotopy α_t by a piecewise linear homotopy $\widehat{\alpha}_t$ such that $\omega_0 + d\widehat{\alpha}_t$ is a homotopy of symplectic forms. Hence we may assume from the very beginning that our original homotopy α_t is linear: $\alpha_t = t\alpha$.

12.1.3. (Piecewise primitive approximation) *Let $t\alpha$, $t \in [0,1]$, be a linear homotopy of 1-forms. Then there exists a piecewise primitive homotopy $\widetilde{\alpha}_t$, $\widetilde{\alpha}_1 = \alpha$, which is arbitrarily C^1-close to $t\alpha$ in the following sense:*

$$||\, t\alpha - \widetilde{\alpha}_t||_{C^1} < \varepsilon \text{ for all } t \in [0,1]\,.$$

In particular, if $\omega + td\alpha$ is a homotopy of symplectic forms then $\omega_0 + d\widetilde{\alpha}_t$ is a homotopy of symplectic forms as well.

Proof of 12.1.3. We will consider the case when V is a compact manifold; the non-compact case can then be treated by a compact exhaustion of V.

Note that it is sufficient to construct an arbitrary piecewise primitive homotopy β_t such that $\beta_0 = 0$ and $\beta_1 = \alpha$; then for sufficiently big N the homotopy $\widetilde{\alpha}_t$ such that

- $\widetilde{\alpha}_{t_i} = t_i\alpha$ for $t_i = i/N$, $i = 0, \dots, N$, and
- $\widetilde{\alpha}_{t_i+\tau} = \widetilde{\alpha}_{t_i} + (1/N)\beta_{N\tau}$, $\tau \in [0, 1/N]$

will be arbitrarily C^1-close to $t\alpha$.

In order to construct β_t it is sufficient to find a decomposition

$$\alpha = \beta_1 + \dots + \beta_L$$

where all β_i are primitive 1-forms.

Using a partition of unity we can reduce to the case when $V = \mathbb{R}^n$ or $\mathbb{R}^n_+$ (half-space) and the 1-form $\alpha = r_1dx_1 + \dots + r_ndx_n$ is supported in a compact set $C \subset V$. Taking a cut-off function $\theta : V \to \mathbb{R}$ which is equal to 1 on C and equal to 0 outside a bigger compact set C' we can rewrite α as $r_1ds_1 + \dots + r_nds_n$, where all the functions $s_i = \theta x_i$, $i = 1, \dots, n$, are compactly supported. $\square$

The above proof also shows the following fact which we will need in Section 12.3 below.

12.1.4. *Any 1-form α on a manifold V can be presented as a sum $\sum_1^N \beta_i$ of primitive forms such that for all $i = 1, \dots, N$ we have $||\beta_i||_{C^1} \le C||\alpha||_{C^1}$ where the number N and the constant C depend only on the ambient manifold V. Moreover, this decomposition can be done in such a way that*

$$\bigcup_1^N \operatorname{Supp} \beta_i \subset \operatorname{Supp} \alpha\,.$$

Now in order to complete the proof of Proposition 12.1.2 we just need to explain how to realize a homotopy of symplectic forms $\omega_0 + t\,dr \wedge ds$ where $r\,ds$ is a primitive 1-form (and, in particular, r and s are compactly supported), by a C^0-small isotopy of symplectic embeddings $V \to (W, \omega_W)$. The following lemma gives us a key to the realization.

12.1.5. (Symplectic twisting) *Let (V, ω) be a symplectic manifold and let D^2_ε denote the disc of radius ε in the standard symplectic plane $(\mathbb{R}^2, \eta = dx \wedge dy)$. Then for any primitive form $r\,ds$ and any $\varepsilon > 0$ there exists a section $\Phi : V \to V \times D^2_\varepsilon$ of the trivial bundle $V \times D^2_\varepsilon \to V$ such that $\Phi^*(\omega \oplus \eta) = \omega + dr \wedge ds$.*

Proof of 12.1.5. The image of the map $\phi = (r, s) : V \to \mathbb{R}^2$ is contained in a disc D^2_R of radius $R > 0$. Take an area preserving map $\tau = \tau_{R,\varepsilon} : D^2_R \to D^2_\varepsilon$ (which, of course, has to be a multiple covering). Then the map

$$\varphi = \tau_{R,\varepsilon} \circ \phi : V \to D^2_\varepsilon$$

induces from the area form $\eta = dx \wedge dy$ on D^2_ε the form $dr \wedge ds$ on V. Hence the corresponding section

$$\Phi : V \to V \times D^2_\varepsilon\,, \quad \Phi(x) = (x, \varphi(x))\,,\ x \in V\,,$$

satisfies the equation $\Phi^*(\omega \oplus \eta) = \omega + dr \wedge ds$. □

We continue the proof of Lemma 12.1.2. Let $U \subset V$ be a coordinate chart which contains the support of the form $r\,ds$. In view of the symplectic neighborhood theorem (see Theorem 9.3.2 above) there exists a positive ε such that the embedding $h_0|_U : U \to W$ extends to an isosymplectic embedding

$$\widehat{h}_0 : (U \times D^2_\varepsilon \times D^{q-n-2}_\varepsilon, \widehat{\omega}_0 = \omega_0 \oplus \eta_2 \oplus \eta_{q-n-2}) \to (W, \omega_W)$$

onto a small neighborhood of $h_0(U)$ in W. Here D^s_ε denotes the ball of radius ε in $\mathbb{R}^s$ and the form η_s is the restriction to D^s_ε of the standard symplectic form on $\mathbb{R}^s$. According to Lemma 12.1.5 there exists a section $\Phi : U \to U \times D^2_\varepsilon$ such that $\Phi^*(\omega_0 \oplus \eta_2) = \omega_0 + dr \wedge ds$. We denote by $\widehat{\Phi}$ the map

$$\Phi \times 0 : U \to \left(U \times D^2_\varepsilon\right) \times D^{q-n-2}_\varepsilon\,,$$

and then define the required isotopy of embeddings $V \to W$ by the formula

$$h_t(x) = \begin{cases} \widehat{h}_0(t\widehat{\Phi}(x)), & x \in U; \\ \widehat{h}_0(x), & \text{otherwise.} \end{cases}$$

This finishes off the proof of Lemma 12.1.2, and together with it the proof of Theorem 12.1.1 in the open case. □

Proof of 12.1.1 in the closed case. Note that the above proof works also in the extension form when the map f_0 is already isosymplectic near a subpolyhedron $A \subset V$ of codimension > 1. Of course, the cohomological condition in this case should refer to the relative cohomology class:

$$[\omega_0]_{H^2(V,A)} = [f_0^* \omega_W]_{H^2(V,A)} \, .$$

Thus to finish the proof in the closed case we just need to explain how to extend the isotopy to a simplex Δ^n of top dimension $n = \dim V$. But in this case the condition $\operatorname{codim} V \geq 4$ allows us to apply the usual microextension trick, multiplying the simplices Δ^n by a small disk $(D^2_\varepsilon, \eta_2)$, and thus reducing the problem to the open case. Note that the C^0-approximation near the core $\Delta^n \times 0 \subset \Delta^n \times D^2_\varepsilon$ provides the C^0-approximation on Δ^n and hence on V. There is, however, a minor problem with this microextension argument: when applying the relative version of Step 1 to the products $\Delta^n \times D^2_\varepsilon$ we need to have the relative cohomological condition

$$[\omega_0]_{H^2(\Delta^n, \partial\Delta^n)} = [f_0^* \omega_W]_{H^2(\Delta^n, \partial\Delta^n)}$$

for each top-dimensional simplex $\Delta^n \subset V$ instead of one global condition

$$[\omega_0]_{H^2(V)} = [f_0^* \omega_W]_{H^2(V)} \, .$$

This problem appears only for $n = 2$ because $H^2(\Delta^n, \partial\Delta^n) = 0$ for $n > 2$. The following lemma completes the proof in the case $n = 2$.

12.1.6. (Localization of the global cohomological condition) *Let (V, ω_V) and (W, ω_W) be symplectic manifolds of dimension $n = 2$ and $q = 2l > n$, respectively. Let $f_0 : V \to W$ be an embedding such that*

- *f_0 is isosymplectic in a neighborhood $\mathcal{O}p\, K^1$ of the 1-skeleton K^1 of a triangulation τ of V and*
- *$f_0^* \omega_W - \omega_V = d\alpha$*

Then there exists an isotopy $f_t : V \to W$, $t \in [0,1]$, such that

- *all the embeddings f_t are isosymplectic on $\mathcal{O}p\, K^1$ and*
- *$\int_{\Delta^2} f_1^* \omega_W = \int_{\Delta^2} \omega_V$ for all 2-simplices Δ^2 of τ.*

Proof. Take any 1-simplex $\sigma \subset K^1$. The embedding $f_0|_\sigma$ is isotropic, and hence its image has the standard symplectic neighborhood (see 9.3.2). Hence there are Darboux coordinates $(x_1, y_1, \dots, x_l, y_l)$ for the symplectic form $\omega_W|_{\mathcal{O}p\, f_0(\sigma)}$ such that the image $f_0(\sigma)$ coincides with the interval

$$I = \{0 \leq x_1 \leq 1,\, y_1 = x_2 = \dots = x_l = y_l = 0\} \, .$$

Set $A_\sigma = \int\limits_\sigma \alpha$. Take an integer N of the same sign as A_σ, choose a smooth function $\theta : [0,1] \to [0, 2\pi N]$ such that $\theta = 0$ near 0 and $\theta = 2\pi N$ near 1, and for a sufficiently small $\varepsilon > 0$ define an isotopy $h_t^\sigma : I \to \mathcal{O}p\, I$, $t \in [0,1]$, by the formula

$$h_t^\sigma(x_1) = (x_1, 0,\, t\varepsilon(1 - \cos\theta(x_1)),\, t\varepsilon \sin\theta(x_1), 0, \dots, 0) \, .$$

Note that the isotopy h_t^σ is fixed near ∂I and

$$\int\limits_{h_t^\sigma(\sigma)} \beta = \pi N \varepsilon^2 t\,,$$

where $\beta = \sum\limits_1^q x_i dy_i$ is the primitive of the symplectic form $\omega_W|_{\mathcal{O}p\, I}$. In particular, for $\varepsilon = \sqrt{\frac{A_\sigma}{N\pi}}$ we get $\int\limits_{h_t^\sigma(\sigma)} \beta = tA_\sigma$, and choosing the absolute value of N sufficiently large we can make the isotopy h_t^σ be arbitrarily C^0-close to the inclusion $I \hookrightarrow \mathcal{O}p\, I$. Now we define an isotopy $f_t : K^1 \to W$ by setting $f_t|_\sigma = h_t^\sigma \circ f_0|_\sigma$ for each 1-simplex $\sigma \subset K^1$. Again using 9.3.2 we can extend f_t to an isotopy defined on the whole V keeping it (ω_V, ω_W)-isosymplectic on $\mathcal{O}p\, K^1$. Then the embedding f_1 will satisfy the cohomological condition

$$\int_{\Delta^2} f_1^* \omega_W = \int_{\Delta^2} \omega_V$$

for all 2-simplices Δ^2 of the triangulation. □

◀ **Remark.** Using the parametric versions of Theorem 4.5.1 and Lemma 12.1.2 we can similarly prove the *parametric* version of Theorem 12.1.1. ▶

12.2. Equidimensional isosymplectic immersions

Lemma 12.1.2 implies, in particular, that one can realize any exact homotopy $\omega_t = \omega + d\alpha_t$ of symplectic forms on (V, ω) by an arbitrarily C^0-small isotopy of embeddings $V \to (V \times \mathbb{R}^2, \omega \oplus dx \wedge dy)$. As we already mentioned, if V is closed then Moser's theorem 9.3.2 allows to substitute $V \times \mathbb{R}^2$ by V at the expense of C^0-approximation. The following proposition shows that for compact manifolds with non-empty boundary one can substitute $V \times \mathbb{R}^2$ by $(\widetilde{V} \times \mathbb{R}, \pi^*\widetilde{\omega})$ where $(\widetilde{V}, \widetilde{\omega})$ is any open symplectic manifold which contains (V, ω) as an equidimensional symplectic submanifold, and π is the projection $\widetilde{V} \times \mathbb{R} \to V$. As in the case of Moser's theorem, it can be done only at the expense of C^0-approximation.

12.2.1. (V. Ginzburg, [Gi97]) *Let $(\widetilde{V}, \widetilde{\omega})$ be a symplectic manifold without boundary and $V \subset \widetilde{V}$ a full-dimensional compact submanifold with boundary. Let $\omega_t = \omega + d\alpha_t$, $t \in [0,1]$, be a family of symplectic forms on V such that $\omega = \omega_0 = \widetilde{\omega}|_V$. Then there exists an isotopy $f_t : V \to \widetilde{V} \times \mathbb{R}^1$, $t \in [0,1]$, such that f_0 is the inclusion*

$$V \hookrightarrow \widetilde{V} = \widetilde{V} \times 0 \hookrightarrow \widetilde{V} \times \mathbb{R}$$

and $f_1^*(\widetilde{\omega}\oplus 0)=\omega_1$ *where* $\widetilde{\omega}\oplus 0=\pi^*\widetilde{\omega}$ *is the pull-back of* $\widetilde{\omega}$ *under the projection* $\widetilde{V}\times\mathbb{R}\to\widetilde{V}$.

◀ **Remarks**

1. The parametric version of the theorem is also true and implies, in particular, that one can choose f_t such that $f_t^*(\widetilde{\omega}\oplus 0)=\omega_t$ for all $t\in[0,1]$.

2. Any diffeomorphism of the form $(x,t)\mapsto(x,h(x,t))$, $x\in\widetilde{V}$, $t\in\mathbb{R}$, preserves the form $\widehat{\omega}$, and hence the image of the isotopy f_t can be shrunk into an arbitrarily small neighborhood of the 0-section $\widetilde{V}\times 0\subset\widetilde{V}\times\mathbb{R}$. However, the isotopy f_t cannot, in general, be chosen C^0-small. ▶

Proof. As in the proof of Lemma 12.1.2 we can approximate α_t by a *piecewise primitive* homotopy $\widetilde{\alpha}_t$. Therefore, it is sufficient to explain how to realize the homotopy of symplectic forms $\omega_0+tdr\wedge ds$, $t\in[0,1]$, by an isotopy of embeddings $f_t:V\to(\widetilde{V},\widetilde{\omega})$. Here $\omega_0=f_0^*(\widetilde{\omega}\oplus 0)$ and the embedding f_0 *is not required* to be the original inclusion $i:V\hookrightarrow\widetilde{V}\times 0\hookrightarrow\widetilde{V}\times\mathbb{R}$. However, the condition $\omega_0=f_0^*(\widetilde{\omega}\oplus 0)$ implies that f_0 is transversal to the (one-dimensional) characteristic foliation $\mathcal{F}_{\widetilde{\omega}\oplus 0}$ of the 2-form $\widetilde{\omega}\oplus 0$ by the fibers of the projection $\widetilde{V}\times\mathbb{R}\to\widetilde{V}$.

Extend the embedding $f_0:V\to\widetilde{V}\times\mathbb{R}$ to an embedding $f_0':V'\to\widetilde{V}\times\mathbb{R}$ of a neighborhood $V'=\mathcal{O}p_{\widetilde{V}}V\subset\widetilde{V}$. We assume that the extension is still transversal to the characteristic foliation $\mathcal{F}_{\widetilde{\omega}\oplus 0}$ on $\widetilde{V}\times\mathbb{R}$. Set

$$\omega_0'=(f_0')^*(\widetilde{\omega}\oplus 0)\,.$$

The embedding f_0' can be extended to a fiber preserving embedding

$$F:V'\times\mathbb{R}\to\widetilde{V}\times\mathbb{R}$$

such that $F|_{V'\times 0}=f_0'$, so that we automatically have

$$F^*(\widetilde{\omega}\oplus 0)=\omega_0'\oplus 0\,,$$

where $\omega_0'\oplus 0$ is the pull-back of ω_0' under the projection $V'\times\mathbb{R}\to V'$.

The section

$$\Phi=(r,s):V\to V\times\mathbb{R}^2\hookrightarrow V'\times\mathbb{R}^2$$

induces the form $\omega_0+dr\wedge ds$ from $\omega_0'\oplus dx\wedge dy$. Take a smooth function $H:V'\times\mathbb{R}\to\mathbb{R}$ which has compact support and such that

$$H(v,r(v))=s(v),\ v\in V\,.$$

Then its graph

$$\Gamma_H=\{(v,t,H(v,t))|\ v\in V',\ t\in\mathbb{R}\}\subset V'\times\mathbb{R}^2$$

contains the image $\Phi(V)$. The crucial observation here is that because the hypersurface Γ_H is graphical, the characteristic foliation on it is diffeomorphic to the foliation by the fibers of the trivial fibration $V' \times \mathbb{R} \to V'$, which is the characteristic foliation of the form $\widehat{\omega}'_0$. In other words, there exists a diffeomorphism $\Psi : \Gamma_H \to V' \times \mathbb{R}$ such that

- $\Psi = \mathrm{Id}$ on $V' \times 0$ and
- $\Psi^*(\omega'_0 \oplus 0) = \Omega$, where $\Omega = (\omega'_0 \oplus dx \wedge dy)|_{\Gamma_H}$.

Moreover, according to Remark 2 preceding this proof we can assume that the image $\Psi(\Phi(V))$ is contained in an arbitrarily small neighborhood of the 0-section $V' \times 0$. Hence the embedding

$$f_1 : (V, \omega_0 + dr \wedge ds) \xrightarrow{\Phi} (\Gamma_H, \Omega) \xrightarrow{\Psi} (V' \times \mathbb{R}, \omega'_0 \oplus 0) \xrightarrow{F} (\widetilde{V} \times \mathbb{R}, \widetilde{\omega} \oplus 0)$$

satisfies the equation $f_1^*(\widetilde{\omega} \oplus 0) = \omega_0 + dr \wedge ds$. We can make all the construction depending on t by taking $t\, dr \wedge ds$ instead of $dr \wedge ds$. In that case $\Phi_t = (\sqrt{t}r, \sqrt{t}s)$, and we choose a family of functions $H : V' \times \mathbb{R} \to \mathbb{R}$ such that

$$H(v, \sqrt{t}\, r(v)) = \sqrt{t}\, s(v),\ v \in V\,.$$

It give us a family of diffeomorphisms $\Psi_t : \Gamma_{H_t} \to V' \times \mathbb{R}$ and hence a homotopy of embeddings $f_t = F \circ \Psi_t \circ \Phi_t : V \to \widetilde{V} \times \mathbb{R}$ such that

$$f_t^*(\widetilde{\omega} \oplus 0) = \omega_0 + t\, dr \wedge ds\,.$$

Note that if the primitive form $r\, ds$ were supported in $V \setminus \partial V$ then one could apply the previous construction without extending V to V'. Alternatively, one could just apply Moser's theorem 9.3.2. □

◀ **Remarks**

1. Note that our construction automatically realizes the approximating piecewise primitive homotopy $\widetilde{\alpha}_t$ by an isotopy f_t such that

$$f_t^*(\widehat{\omega}) = \omega_0 + \widetilde{\alpha}_t \ \text{ for } \textit{all} \ \ t \in [0,1]\,.$$

2. The diffeomorphism Ψ is the key ingredient in the proof. In the case when the function H is supported in $V \times \mathbb{R}$, any fiber of the characteristic foliation $\mathcal{F}_\Omega$ on Γ_H intersects the image $\Phi(V)$ only once (or does not intersect $\Phi(V)$ at all). However, in the general case the characteristics on Γ_h may intersect the image $\Phi(V)$ many times because V' is strictly bigger than V. This is the reason why the projection of the constructed embedding $f_1 : V \to \widetilde{V} \times \mathbb{R}$ to $\widetilde{V}$ may become an (equidimensional) *immersion*, rather than an embedding already after the realization of the first segment of the approximating primitive homotopy $\widetilde{\alpha}_t$.

3. Ψ cannot be made C^0-small. Neither the application of symplectic twisting (Lemma 12.1.5), nor the smallness of the supports of the primitive forms can salvage the C^0-approximation property for f_t (why?).

4. The proof works also for a closed manifold V and an exact homotopy of symplectic forms on V. In this case the result, of course, is not new: it is just a version of Moser's theorem 9.3.2. ▶

Proposition 12.2.1, or rather its 1-parametric version, has as a corollary the following version of Moser's theorem 9.3.2 for manifolds with boundary, which is formulated as an Exercise in Gromov's book [**Gr86**], p.335.

12.2.2. (Realization of an exact homotopy of symplectic forms by a homotopy of equidimensional immersions) *Let ω_t, $t \in [0,1]$, be a family of symplectic forms on a compact manifold V with boundary. Suppose that all these forms belong to the same cohomology class from $H^2(V)$. Let $(\widetilde{V}, \widetilde{\omega})$ be a symplectic manifold without boundary which contains V as its equidimensional submanifold, so that $\widetilde{\omega}|_V = \omega_0$. Then there exists a regular homotopy $f_t : V \to \widetilde{V}$ such that f_0 is the inclusion $V \hookrightarrow \widetilde{V}$ and*

$$f_t^* \widetilde{\omega} = \omega_t, \ \ t \in [0,1] \, .$$

12.3. Isocontact embeddings

Let us recall that for a contact manifold (W, ξ_W) a map $f : V \to W$ is called *contact* if it induces a contact structure on V. A homomorphism $F : V \to W$ is called contact if it is transversal to ξ_W and the intersection $F(TV) \cap \xi_W$ consists of symplectic subspaces of ξ_W with respect to the conformal symplectic structure $\mathrm{CS}(\xi_W)$. If the manifold V itself has a contact structure then we also consider *isometric contact* or *isocontact* maps $f : (V, \xi_V) \to (W, \xi_W)$ which induce on V the given structure ξ_V. A monomorphism $F : TV \to TW$ is called *isocontact* if $\xi_V = F^{-1}(\xi_W)$ and F induces a conformally symplectic map $\xi_V \to \xi_W$ with respect to the conformal symplectic structures $\mathrm{CS}(\xi_V)$ and $\mathrm{CS}(\xi_W)$.

◀ **Remark.** As in the symplectic case the term "isometric contact" was used by G. D'Ambra, see [**DA00**], in a different sense. Namely for two contact manifolds endowed with compatible CR-structures, and therefore Riemannian metrics on the contact distributions, she considered the problem of finding immersions which are simultaneously isocontact and isometric with respect to the given Riemannian metrics. She proved in [**DA00**] an h-principle type result which is a mixture of Theorem 12.3.1 and Nash-Kuiper's C^1-isometric immersion theorem, see Theorem 21.2.1 below. ▶

As in the symplectic case the contact condition is open, while the isocontact one is not. Hence, for an open V Theorem 10.3.2 implies the parametric h-principle for contact *immersions*, while Theorem 4.5.1 yields a theorem about directed contact *embeddings*. We turn now to the problem of *isocontact* embeddings. Here is an analog of Gromov's symplectic embedding theorem 12.1.1 for the contact case.

12.3.1. (Isocontact embeddings) *Let (V, ξ_V) and (W, ξ_W) be contact manifolds of dimension $n = 2m + 1$ and $q = 2l + 1$, respectively. Suppose that the differential $F_0 = df_0$ of an embedding $f_0 : (V, \xi_V) \to (W, \xi_W)$ is homotopic (via a homotopy of monomorphisms $F_t : TV \to TW$, $\operatorname{bs} F_t = f_0$) to an isocontact monomorphism $F_1 : TV \to TW$.*

- **Open case.** *If $n \leq q - 2$ and the manifold V is open then there exists an isotopy $f_t : V \to W$ such that the embedding $f_1 : V \to W$ is isocontact and the differential df_1 is homotopic to F_1 through isocontact homomorphisms. Moreover, given a core $V_0 \subset V$ one can choose the isotopy f_t to be arbitrarily C^0-close to f_0 near V_0.*
- **Closed case.** *If $n \leq q-4$ then the above isotopy f_t exists even if V is closed. Moreover, one can choose the isotopy f_t to be arbitrarily C^0-close to f_0.*

Proof. We will consider only the case of open manifolds. The reduction of the closed case to the open one via the microextension trick is straightforward for contact manifolds because one does not have an additional cohomological condition to worry about.

As in the symplectic case the construction of the isotopy f_t proceeds in two steps.

Step 1. The homotopy F_t gives us a homotopy $G_t = GF_t$ between the tangent lift $G_0 = Gdf_0$ of f_0 and a map $G_1 : V \to A_{\text{cont}} \subset \text{Gr}_n W$. The set $A_{\text{cont}} \subset \text{Gr}_n W$ is open, and hence according to Theorem 4.5.1 there exists an isotopy $\widetilde{f}_t : V \to W$, $\widetilde{f}_0 = f_0$, such that $\widetilde{f}_1 : V \to W$ is a *contact embedding* whose tangential $Gd\widetilde{f}_1$ is homotopic to G_1. According to 4.5.2 one can additionally arrange that the differential $d\widetilde{f}_1$ and the isocontact homomorphism F_1 are homotopic via a homotopy Ψ_t for which $G\Psi_t(V) \subset A_{\text{cont}}$. This fact and Theorem 10.3.2 imply the existence of a family of contact structures $\widetilde{\xi}_t$ which connects the contact structures ξ_V and $\widetilde{f}_1^* \xi_W$.

Step 2. Similarly to the symplectic case the statement of Theorem 12.3.1 is a corollary of the following lemma

12.3.2. *Let V be a compact manifold with boundary and $\xi_t, t \in [0, 1]$, be a family of contact structures on V. Let $f : (V, \xi_0) \to (W, \xi_W)$ be an isocontact* *embedding. Then there exists a contact isotopy* $f_t : V \to W$, $t \in [0,1]$, *such that* $f_0 = f$ *and* f_1 *is an isocontact embedding* $(V, \xi_1) \to (W, \xi_W)$.

◀ **Remark.** The parametric version of 12.3.2 is also true with essentially the same proof. Hence, under the assumptions of 12.3.2 one can arrange that the isotopy f_t consists of isocontact embeddings $(V, \xi_t) \to (W, \xi_W)$ *for all* $t \in [0,1]$. ▶

To prove Lemma 12.3.2 we will need three sublemmas 12.3.3, 12.3.4 and 12.3.5.

Let U be a compact domain with piecewise smooth boundary in a contact manifold (V, ξ). We will say that U is *contactly contractible* if there exists a contact vector field X on V which is inward transversal to the boundary of U and such that given a contact form α the flow $X^t : V \to V$ contracts α when $t \to +\infty$. In other words $(X^t)^*\alpha \underset{t\to+\infty}{\longrightarrow} 0$. Note that this property is independent of the choice of the contact from α.

◀ **Problem.** Find an effective criterium for a domain to be contactly contractible. In particular, are geometrically convex domains in the standard contact space

$$(\mathbb{R}^{2m+1}, \zeta_m = \{dz + \sum_1^m x_j dy_j - y_j dx_j = 0\})$$

contactly contractible? ▶

The next lemma lists a few simple examples of contactly contractible domains which we will need below in the proof of 12.3.2.

12.3.3. *Let the space* $\mathbb{R}^n = \mathbb{R}^{2m+1}$ *be endowed with the structure* $\zeta_m = \{dz + \sum\limits_1^m x_j dy_j - y_j dx_j = 0\}$. *Then*

- **(a)** *the Euclidean ball* $D = D^n_R(0) \subset \mathbb{R}^n$ *centered at the origin is contactly contractible;*
- **(b)** *semi-balls* $D^n_{R,l} = D^n_R \cap \{l \geq 0\} \subset \mathbb{R}^n$, *where* $l : \mathbb{R}^n \to \mathbb{R}$ *is a linear function are contactly contractible;*
- **(c)** *if* U *is contactly contractible then for any* C^1*-small diffeomorphism* $f : \mathbb{R}^n \to \mathbb{R}^n$ *the domain* $f(U)$ *is contactly contractible as well.*

Proof. The statement 12.3.3 c) is obvious. The contact vector field

$$X = -\sum_1^m \left(x_i \frac{\partial}{\partial x_i} + y_i \frac{\partial}{\partial y_i} \right) - 2z \frac{\partial}{\partial z}$$

does the contracting job in case a). The contact structure ζ_m is invariant under rotations around the z-axis, and hence any semi-ball centered at the origin is contactomorphic to a semiball $D^n_{R,l}$ with $l = az + bx_1$. If $a \neq 0$ then the corresponding contracting contact vector field X can be chosen equal to

$$-\sum_1^m \left(x_i \frac{\partial}{\partial x_i} + y_i \frac{\partial}{\partial y_i} \right) - 2(z+c)\frac{\partial}{\partial z},$$

where $\frac{R}{2} < c < R$. If $a = 0$ then we can assume $l = x_1$ and in this case take the contact vector field

$$X = (\varepsilon - x_1)\frac{\partial}{\partial x_1} - \sum_2^m x_i \frac{\partial}{\partial x_i} - \sum_1^m y_j \frac{\partial}{\partial y_j} - (2z + \varepsilon y_1)\frac{\partial}{\partial z},$$

$0 < \varepsilon < \frac{R}{R+1}$, which contracts $D^n_{R,l}$. □

12.3.4. *Let U be a contact contractible domain in a contact manifold $(V, \xi = \{\alpha = 0\})$. Suppose that the form $\beta = \alpha + r\,ds$ is contact for functions $r, s : \mathbb{R} \to \mathbb{R}$. Then for any $\varepsilon > 0$ there exists an isocontact embedding*

$$f : (V, \{\beta = 0\}) \to \big(V \times D^2_\varepsilon, \{\alpha \oplus xdy = 0\}\big)$$

If the functions r, s vanish on $\mathcal{O}p\,A \subset U$, where A is a closed subset of the boundary ∂U then on $\mathcal{O}p\,A$ the embedding f can be chosen equal to the inclusion $U = U \times 0 \hookrightarrow U \times D^2_\varepsilon$.

Proof. The section $F : U \to U \times \mathbb{R}^2$, $F(u) = (u, r(u), s(u))$, $u \in U$, is isocontact with respect to the contact structures defined by the contact forms $\beta = \alpha + r\,ds$ on U and $\alpha \oplus x\,dy$ on $U \times \mathbb{R}^2$. Let X be the contracting contact vector field for the domain U and $K = K_X$ its contact Hamiltonian (see Chapter 9.7 above). By the definition of contact contractibility we have $\lim\limits_{t \to +\infty} h_t = 0$ where the function h_t is defined by the equation $\left(X^t\right)^* \alpha = h_t \alpha$ and $X^t : U \to U$, $t \in \mathbb{R}_+$, denotes the forward contact flow of the contact vector field X. Let $\widetilde{U} \subset U$ be a slightly smaller domain with piecewise smooth boundary such that

- X is inward transversal to $\partial \widetilde{U}$;
- $U \setminus \widetilde{U} = \mathcal{O}p\,A$;
- the functions r, s vanish on $U \setminus \widetilde{U}$.

Take a function $\theta : U \to \mathbb{R}_+$ which is equal to 1 on $\widetilde{U}$ and equal to 0 on $\mathcal{O}p\,A$. Let $\widetilde{X}$ be the contact vector field defined by the contact Hamiltonian θK. Let $\widetilde{X}^t : U \to U$, $t \in \mathbb{R}_+$, denotes the forward contact flow of the contact vector field $\widetilde{X}$, and the family of functions $\widetilde{h}_t : U \to \mathbb{R}^+$ be defined by the formula

$$\left(\widetilde{X}^t\right)^* \alpha = \widetilde{h}_t \alpha \,.$$

Then $\widetilde{h}_t|_{\widetilde{U}} = h_t|_{\widetilde{U}} \underset{t\to+\infty}{\to} 0$ and $\widetilde{h}_t|_{\mathcal{O}p\, A} \equiv 1$. For each $t \geq 0$ we define the map $\Psi_t : U \times \mathbb{R}^2 \to U \times \mathbb{R}^2$ by the formula

$$\Psi_t(u,x,y) = \left(\widetilde{X}^t(u), \sqrt{\widetilde{h}_t(u)}x, \sqrt{\widetilde{h}_t(u)}y\right), \quad (u,x,y) \in U \times \mathbb{R}^2 \,.$$

Then

$$\Psi_t^*(\alpha \oplus x\,dy) = \widetilde{h}_t(\alpha \oplus x\,dy) \,,$$

and hence Ψ_t preserves the contact structure $\{(\alpha \oplus x\,dy) = 0\}$. Therefore $\Psi_t \circ F$ is a family of isocontact embeddings

$$(U, \{\beta = 0\}) \to \left(U \times \mathbb{R}^2, \{\alpha \oplus x\,dy = 0\}\right) \,.$$

Let us denote by g_t the composition $\pi \circ \Psi_t \circ F : U \to \mathbb{R}^2$ where $\pi : V \times \mathbb{R}^2 \to \mathbb{R}^2$ is the projection to the second factor. Then $\lim\limits_{t\to\infty} g_t = 0$. Indeed, we have $g_t(u) = \widetilde{h}_t(u) g_0(u)$. If $u \in \widetilde{U}$ then $\widetilde{h}_t(u) = h_t(u) \underset{t\to\infty}{\longrightarrow} 0$. If $u \in U \setminus \widetilde{U}$ then $g_0(u) = (r(u), s(u)) = 0$, and hence $g_t(u) \equiv 0$ for all $t \geq 0$. Thus for any $\varepsilon > 0$ there exists $T > 0$ such that $\Psi_t \circ F(U) \subset U \times D^2_\varepsilon$ for $t \geq T$. Besides, we have $\Psi_t \circ F(u) = (u,0,0)$ for $u \in \mathcal{O}p\, A$ and all $t \geq 0$. Hence $f = \Psi_T \circ F$ is the required isocontact embedding $(U, \{\beta = 0\}) \to (U \times \mathbb{R}^2, \{\alpha \oplus x\,dy = 0\})$. ☐

12.3.5. *Let $\alpha_t, t \in [0,1]$, be a family of contact forms for the family of contact structures ξ_t on V, as in Lemma 12.3.2. Then there exists a sequence of primitive 1-forms $\beta_i = r_i ds_i,\ i = 1, \dots, \beta_M$, such that*

- $\alpha_1 = \alpha_0 + \sum\limits_1^M \beta_j$ *;*
- *for each $k = 0, \dots, M$ the form $\alpha^{(k)} = \alpha_0 + \sum\limits_1^k \beta_j$ is contact;*
- *for each $j = 1, \dots, M$ the functions r_j and s_j are supported in a domain which is contactly contractible for the contact structure $\xi^{(j-1)} = \{\alpha^{(j-1)} = 0\}$ and homeomorphic to a ball.*

Proof. For each $t \in [0,1]$ there exists a covering of V by domains U_t^i, $i = 1, \dots, K$, such that each domain (U_t^i, ξ_t) is isomorphic either to the ball $B_R(0)$ or a semiball $B_{R,l}(0)$ in the standard contact space

$$\left(\mathbb{R}^{2m+1}, \zeta_m = \{dz + \sum_1^m x_j dy_j - y_j dx_j = 0\}\right) .$$

In view of compactness of V, Lemma 12.3.3 c) implies the existence of a $\delta > 0$ such that each domain $U_t^i,\ t \in [0,1],\ i = 1, \dots K$, is contactly contractible for *all* contact structures ξ_τ with $\tau \in [t-\delta, t+\delta]$. For each $t \in [0,1]$ let us choose a partition of unity $\sum\limits_1^L \sigma_t^{(i)} = 1$ subordinated to the covering $U_t^1, \dots, U_t^K$. Choose an integer N, set $t_i = \frac{i}{N}$, $i = 0, \dots, N$, and for $i = 0, \dots, N-1$, $j = 0, \dots, L$ consider 1-forms

$$\alpha^{(ij)} = \alpha_{t_i} + \sum_{k=1}^{j} \beta^{(kj)}, \quad \text{where} \quad \beta^{(kj)} = \sigma_{t_i}^{(k)} \left(\alpha_{t_{i+1}} - \alpha_{t_i}\right) .$$

If N is chosen sufficiently large then all the forms $\alpha^{(ij)}$ are contact and, moreover, for a fixed $i = 0, \dots, N$ the domains $U_{t_i}^1, \dots, U_{t_i}^K$ are contactly contractible with respect to the contact structures defined by *all the contact forms* $\alpha^{(ij)}$, $j = 0, \dots, L$. Lemma 12.1.4 allows us to decompose each form $\beta^{(ij)}$ as a sum $\beta^{(ij)} = \sum\limits_k r_k^{(ij)} ds_k^{(ij)}$, where the functions $r_k^{(ij)}, s_k^{(ij)}$ have the same support as $\beta^{(ij)}$,

$$||r_k^{(ij)} ds_k^{(ij)}||_{C^1} \leq C||\beta^{(ij)}||_{C^1},$$

and the constant C depends only on the domain $U_{t_i}^s$ which supports $\sigma_{t_i}^{(k)}$. To avoid the growing number of indices we will assume that the latter decomposition consists of just one term, i.e. $\beta^{(ij)} = r^{(ij)} ds^{(ij)}$.

It remains to order linearly the forms $\beta^{(ij)}$ as

$$\beta^{(01)}, \beta^{(02)}, \dots, \beta^{(0L)}, \beta^{(11)}, \dots, \beta^{(NL)}$$

to obtain the desired sequence of forms $\beta_1, \dots, \beta_M$. □

Proof of 12.3.2. Let $\beta_1, \dots, \beta_M$ be the sequence of forms provided by Lemma 12.3.5 and

$$\alpha^{(k)} = \alpha_0 + \sum_1^k \beta^{(j)}, \ \ k = 0, \dots, M .$$

We will construct inductively a sequence of isocontact embeddings

$$f^{(j)} : (V, \xi^{(j)} = \{\alpha^{(j)} = 0\}) \to (W, \xi_W)\,, \ \ j = 0, \dots, M\,,$$

beginning with $f^{(0)} = f$. Then $f_1 = f^{(M)}$ will be the desired isocontact embedding $(V, \xi_1) \to (W, \xi_W)$.

Suppose that for $j = 0, \dots, M-1$ the embedding $f^{(j)}$ has already been constructed. The contact structures $\xi^{(j)}$ and $\xi^{(j+1)}$ differ only over a domain U with piecewise smooth boundary which is homeomorphic to a ball and which is contactly contractible for the contact structure $\xi^{(j)}$. According to the Contact Neighborhood Theorem 9.5.2 the embedding

$$f^{(j)}|_U : (U, \xi^{(j)}) \to (W, \xi_W)$$

extends for a sufficiently small $\varepsilon > 0$ to an isocontact embedding

$$F^{(j)} : (U \times D^2_\varepsilon \times D^{q-n-2}_\varepsilon, \widetilde{\xi}^{(j)}) \to (W, \xi_W)\,,$$

where the contact structure $\widetilde{\xi}^{(j)}$ is defined by the contact form

$$\alpha^{(j)} \oplus x\,dy \oplus \sum_1^{q-n-2} x_k dy_k\,.$$

If ε is chosen sufficiently small then the image of the embedding $F^{(j)}$ does not intersect $f^{(j)}(V \setminus U)$. Using Lemma 12.3.4 we can construct an isocontact embedding

$$f : (U, \xi^{(j+1)}) \to (U \times D^2_\varepsilon, \{\alpha^{(j)} \oplus x\,dy = 0\}).$$

Moreover, if the functions $r^{(j+1)}$ and $s^{(j+1)}$ vanish on $\mathcal{O}p\,A$ where $A \subset \partial U$, then the embedding f can be chosen to coincide with the inclusion on $\mathcal{O}p\,A$. We combine f with the isocontact inclusion

$$\Big(U \times D^2_\varepsilon, \{\alpha^{(j)} \oplus x\,dy = 0\}\Big) \hookrightarrow \Big(U \times D^2_\varepsilon \times D^{q-n-2}_\varepsilon, \widetilde{\xi}^{(j)}\Big)\,,$$

where $\widetilde{\xi}^{(j)} = \{\alpha^{(j)} \oplus x\,dy \oplus \sum_1^{q-n-2} x_k dy_k = 0\}$, to obtain an isocontact embedding

$$f' : (U,\, \xi^{(j+1)}) \to (U \times D^2_\varepsilon \times D^{q-n-2}_\varepsilon,\, \widetilde{\xi}^{(j)}).$$

Hence the required isocontact embedding $f^{(j+1)} : (V, \xi^{(j+1)}) \to (W, \xi_W)$ can be defined to be equal to $f^{(j)}$ on $V \setminus U$ and equal $F^{(j)} \circ f'$ on U. This finishes off the proof of Lemma 12.3.2, and together with it the proof of Theorem 12.3.1. □

◀ **Remark.** The parametric versions of Theorems 12.1.1 and 12.3.1 are also valid with the same proof. One can also deduce the corresponding results about isosymplectic and isocontact *immersions*. Later in Chapter 16 we will prove the results about isosymplectic and isocontact immersions by a different method which would allow us to get rid of any dimensional restrictions. ▶

◀ **Exercise.** Formulate and prove contact analogs of Theorem 12.2.1 and Corollary 12.2.2. ▶

12.4. Subcritical isotropic embeddings

Let us point out a useful application of Theorems 12.1.1 and 12.3.1.

Let W be either a symplectic or a contact manifold of dimension q, and V a smooth manifold of subcritical dimension, i.e. $\dim V < \left[\frac{\dim W - 1}{2}\right]$. Let $\mathrm{Mono}^{\mathrm{emb}}$ be the space of monomorphisms $TV \to TW$ which cover *embeddings* $V \to W$, and $\mathrm{Mono}^{\mathrm{emb}}_{\mathrm{isot}}$ its subspace which consists of isotropic monomorphisms $F : TV \to TW$, i.e. of monomorphisms which send tangent spaces to V to isotropic subspaces of TW in the symplectic case, and to isotropic subspaces of the contact bundle $\xi \subset TW$ in the contact case. Let $\mathrm{Mono}^{\mathrm{emb}}_{\mathrm{isot}}$ be the space of homotopies

$$\mathrm{Mono}^{\mathrm{emb}}_{\mathrm{isot}} = \{F_t, t \in [0,1] \mid F_t \in \mathrm{Mono}^{\mathrm{emb}},\ F_0 = df_0,\ \ F_1 \in \mathrm{Mono}^{\mathrm{emb}}_{\mathrm{isot}}\}.$$

The space $\mathrm{Emb}_{\mathrm{isot}}$ of isotropic embeddings $V \to W$ can be viewed as a subspace of $\mathrm{Mono}^{\mathrm{emb}}_{\mathrm{isot}}$. Indeed, we can associate with $f \in \mathrm{Emb}_{\mathrm{isot}}$ the homotopy $F_t \equiv df, t \in [0,1]$, from $\mathrm{Mono}^{\mathrm{emb}}_{\mathrm{isot}}$.

12.4.1. (Homotopy principle for subcritical isotropic embeddings) *The inclusion*

$$\mathrm{Emb}_{\mathrm{isot}} \hookrightarrow \mathrm{Mono}^{\mathrm{emb}}_{\mathrm{isot}}$$

is a homotopy equivalence.

The above h-principle also holds in the relative and C^0-dense forms.

Proof. In the symplectic case, any isotropic homomorphism extends in a homotopically canonical way to an isosymplectic homomorphism $T(T^*V) \to TW$, and in the contact case it extends to an isocontact homomorphism $T(J^1(V)) \to TW$, where T^*V and $J^1(V)$ are endowed with the canonical symplectic and contact structures. The dimensional condition ensures in both the symplectic and the contact cases that the dimensions of T^*V and $J^1(V)$ are $\leq \dim W - 2$, and hence Theorems 12.1.1 and 12.3.1 apply. Then we get the required isotropic embeddings as the restrictions to the 0-section of the constructed isosymplectic and isocontact embeddings. □

◀ **Warning.** The analogs of 12.1.1 and 12.3.1 for Lagrangian and Legendrian embeddings are wrong. ▶

Chapter 13

Microflexibility and Holonomic $\mathcal{R}$-approximation

For further applications to Symplectic geometry we will need a generalization of the Holonomic Approximation Theorem which we discuss in Section 13.4 below.

13.1. Local integrability

A differential relation $\mathcal{R} \subset X^{(r)}$ is called *locally integrable* if for any $v \in V$ and any section $F : v \to \mathcal{R}$ there exists a local holonomic extension $\widetilde{F}$ of F, i.e. a holonomic section $\widetilde{F} : \mathcal{O}p\, v \to \mathcal{R}$ such that $\widetilde{F}(v) = F(v)$. In other words, the Cauchy problem with the initial data $(v, F(v))$ has a local solution.

More generally, a differential relation $\mathcal{R} \subset X^{(r)}$ is called *(parametrically) locally integrable* if given a map $h : I^k \to V$ and a family of sections

$$F_p : h(p) \to \mathcal{R}\,,\ p \in I^k\,,$$

there exists a family of local holonomic extension

$$\widetilde{F}_p : \mathcal{O}p\, h(p) \to \mathcal{R}\,,\ \ \widetilde{F}_p(h(p)) = F_p(h(p))\,,\ \ p \in I^k\,.$$

◄ **Exercise.** Prove that the parametric local integrability of $\mathcal{R}$ implies that for any $v \in V$ and any local section $F : \mathcal{O}p\, v \to \mathcal{R}$ there exists a homotopy

$$F^\tau : \mathcal{O}p\, v \to \mathcal{R}\,,\ \tau \in [0,1]\,,$$

such that $F^\tau(v) = F(v)$ for all $\tau \in [0,1]$, F^0 coincides with F and F^1 is holonomic. In other words, the Cauchy problem with the initial data $(v, F(v))$ has a local solution *in any homotopy class of local sections* $\mathcal{O}p\, v \to \mathcal{R}$. ▶

In fact, we need the following stronger *relative* version of parametric local integrability: a differential relation $\mathcal{R} \subset X^{(r)}$ is called *(parametrically) locally integrable* if given a map $h : I^k \to V$, a family of sections

$$F_p : h(p) \to \mathcal{R}\,,\ p \in I^k\,,$$

and a family of local holonomic extensions near ∂I^k

$$\widetilde{F}_p : \mathcal{O}p\ h(p) \to \mathcal{R}\,,\ \ \widetilde{F}_p(h(p)) = F_p(h(p))\,,\ \ p \in \mathcal{O}p\,(\partial I^k)\,,$$

there exists a family of local holonomic extensions

$$\widetilde{F}_p : \mathcal{O}p\, h(p) \to \mathcal{R}\,,\ \ \widetilde{F}_p(h(p)) = F_p(h(p))\,,\ \ \text{for all}\ \ p \in I^k\,,$$

such that for $p \in \mathcal{O}p\,(\partial I^k)$ these new extensions coincide with the original extensions over $\mathcal{O}p\, h(p)$.

In what follows the term *locally integrable* always means this last stronger version of local integrability.

◀ **Exercise.** Prove that the local integrability of $\mathcal{R}$ implies the local h-principle for $\mathcal{R}$ over any point $v \in V$. ▶

◀ **Examples**

1. Any open differential relation is locally integrable, see (the parametric version of) Lemma 3.3.2.

2. The differential relation $\mathcal{R}_{\text{iso}}$ which defines isometric immersions of Riemannian manifolds $(V, g_V) \to (W, g_W)$ is not locally integrable in general.

3. Symplectic and contact stability (Theorems 9.3.2 and 9.5.2) imply that the following closed differential relations of symplectic-geometric origin are locally integrable:

- the differential relation $\mathcal{R}_{\text{isosymp}}$ which defines isosymplectic immersions $(V, \omega_V) \to (W, \omega_W)$;
- the differential relations $\mathcal{R}_{\text{Lag}}$ and $\mathcal{R}_{\text{sub-isotr}}$ which define Lagrangian and subcritical isotropic immersions $V \to (W, \omega_W)$;
- the differential relation $\mathcal{R}_{\text{isocont}}$ which defines isocontact immersions $(V, \xi_V) \to (W, \xi_W)$;
- the differential relations $\mathcal{R}_{\text{Leg}}$ and $\mathcal{R}_{\text{sub-isotr}}$ which define Legendrian and subcritical isotropic immersions $V \to (W, \xi_W)$. ▶

13.2. Homotopy extension property for formal solutions

Let $\mathcal{R} \subset X^{(r)}$ be a differential relation and A a compact subset of V.

13.2.1. (Homotopy extension property for formal solutions) *Let $F : V \to \mathcal{R}$ be a section and $F_A^\tau : \mathcal{O}p\, A \to \mathcal{R}$, $\tau \in [0,1]$, be a homotopy of the section $F_A^0 = F|_{\mathcal{O}p\, A}$. Then F_A^τ extends to a homotopy $F^\tau : V \to \mathcal{R}$ of F.*

Proof. The homotopy F_A^τ is defined over a neighborhood $U \subset V$. Take a continuous function $\delta : V \to \mathbb{R}_+$ with a compact support in U such that $\rho \equiv 1$ in a neighborhood $U' \subset U$ of A. Then set $F^\tau(v) = F(v)$ for $v \in V \setminus U$ and $F^\tau(v) = F_A^{\tau\delta(v)}$ for $v \in U$. □

Similarly, we can prove the following

13.2.2. (Homotopy extension property for formal solutions, relative case) *Let (A, B), where $B \subset A$ be a pair of compact subsets of V, $F_A : \mathcal{O}p\, A \to \mathcal{R}$ a section and $F_B^\tau : \mathcal{O}p\, B \to \mathcal{R}$, $\tau \in [0,1]$, a homotopy of the section $F_B^0 = F_A|_{\mathcal{O}p\, B}$. Then F_B^τ extends to a homotopy $F_A^\tau : \mathcal{O}p\, A \to \mathcal{R}$ of F_A.*

13.3. Microflexibility

The notion of a microflexible differential relation which we introduce in this section roughly corresponds to Gromov's notion of a *microflexible* sheaf, see [**Gr69**] and [**Gr86**].

Let us recall that the term *holonomic homotopy* (or *holonomic deformation*) is used in a sense of homotopy consisting of holonomic sections.

Let $K^m = [-1,1]^m$. For a fixed n and any $k < n$ we denote by θ_k the pair $(K^n, K^k \cup \partial K^n)$. Let, as usual, V be an n-dimensional manifold. A pair $(A, B) \subset V$ is called *θ-pair* or, more precise, θ_k-pair, if (A, B) is diffeomorphic to the standard pair θ_k. [1]

A differential relation $\mathcal{R} \subset X^{(r)}$ is called *k-microflexible* if for any sufficiently small open ball $U \subset V$ and any

- θ_k-pair $(A, B) \subset U$,
- holonomic section $F^0 : \mathcal{O}p\, A \to \mathcal{R}$ and
- holonomic homotopy $F^\tau : \mathcal{O}p\, B \to \mathcal{R}$, $\tau \in [0,1]$, of the section F^0 over $\mathcal{O}p\ B$ which is constant over $\mathcal{O}p\,(\partial B)$

[1] We use the term *θ-pair*, because the shape of the set $K^k \cup \partial K^n$ resembles the letter θ.

there exist a number $\sigma > 0$ and a holonomic homotopy, constant over $\mathcal{O}p\,(\partial A)$,

$$F^\tau : \mathcal{O}p\,A \to \mathcal{R}\,, \quad \tau \in [0,\sigma]\,,$$

which extends the homotopy

$$F^\tau : \mathcal{O}p\,B \to \mathcal{R}\,, \quad \tau \in [0,\sigma]\,.$$

In other words, an initial stage of any holonomic deformation of the section F^0 over $\mathcal{O}p\,B$ which is constant over $\mathcal{O}p\,(\partial B)$ can be extended to a holonomic deformation of F^0 over $\mathcal{O}p\,A$ which is constant over $\mathcal{O}p\,(\partial A)$. If such an extension exists for all $\tau \in [0,1]$ then $\mathcal{R}$ is called *k-flexible*.

More generally, a differential relation $\mathcal{R} \subset X^{(r)}$ is called *parametrically k-microflexible* if for any sufficiently small open ball $U \subset V$ and any families parametrized by $p \in I^m$ of

- θ_k-pairs $(A_p, B_p) \subset U$,
- holonomic sections $F^0_p : \mathcal{O}p\,A_p \to \mathcal{R}$ and
- holonomic homotopies $F^\tau_p : \mathcal{O}p\,B_p \to \mathcal{R}$, $\tau \in [0,1]$, of the sections F^0_p over $\mathcal{O}p\,B_p$ which are constant over $\mathcal{O}p\,(\partial B_p)$ for all $p \in I^m$ and constant over $\mathcal{O}p\,B$ for $p \in \mathcal{O}p\,(\partial I^m)$

there exist a number $\sigma > 0$ and a family of holonomic homotopies

$$F^\tau_p : \mathcal{O}p\,A_p \to \mathcal{R}\,, \quad \tau \in [0,\sigma]\,,$$

which extend the family of homotopies

$$F^\tau_p : \mathcal{O}p\,B_p \to \mathcal{R}\,, \quad \tau \in [0,\sigma]\,,$$

and are constant over $\mathcal{O}p\,(\partial A_p)$ for all $p \in I^m$ and constant over $\mathcal{O}p\;A$ for $p \in \mathcal{O}p\,(\partial I^m)$.

In what follows the term *k-(micro)flexible* always means *parametrically k-(micro)flexible.*

A differential relation $\mathcal{R} \subset X^{(r)}$ is called *(micro)flexible*, if $\mathcal{R}$ is k-(micro)flexible for all $k = 0, \dots, n-1$ where $n = \dim V$.

◀ **Examples**

1. Any open differential relation is microflexible.

2. The differential relation $\mathcal{R}_{\mathrm{hol}} \subset (X^{(r)})^{(1)}$ which defines holonomic sections of $X^{(r)}$ is flexible.

3. The differential relation $\mathcal{R}_{\mathrm{clo}} \subset (\Lambda^p V)^{(1)}$ which defines closed p-forms on V, is k-flexible if $k \neq p$. For $k = p$ the relation $\mathcal{R}_{\mathrm{clo}}$ is neither k-flexible nor k-microflexible.

4. The symplectic and contact stability (Theorems 9.3.2 and 9.5.2) implies that

- the differential relation $\mathcal{R}_{\text{isosymp}}$ which defines isosymplectic immersions $(V, \omega_V) \to (W, \omega_W)$ is k-microflexible if $k \neq 1$.
- the differential relations $\mathcal{R}_{\text{Lag}}$ and $\mathcal{R}_{\text{sub}-\text{isotr}}$ which define Lagrangian and subcritical isotropic immersions $V \to (W, \omega_W)$ are k-microflexible if $k \neq 1$.
- the differential relation $\mathcal{R}_{\text{isocont}}$ which defines isocontact immersions $(V, \xi_V) \to (W, \xi_W)$ is microflexible.
- the differential relations $\mathcal{R}_{\text{Leg}}$ and $\mathcal{R}_{\text{sub}-\text{isotr}}$ which define Legendrian and subcritical isotropic immersions $V \to (W, \xi_W)$ are microflexible.

►

13.4. Theorem on holonomic $\mathcal{R}$-approximation

13.4.1. (Holonomic $\mathcal{R}$-Approximation Theorem) *Let $\mathcal{R} \subset X^{(r)}$ be a locally integrable microflexible differential relation. Let $A \subset V$ be a polyhedron of positive codimension and $F : \mathcal{O}p\, A \to \mathcal{R}$ be a section. Then for arbitrarily small $\delta, \varepsilon > 0$ there exist a δ-small (in the C^0-sense) diffeotopy $h^\tau : V \to V$, $\tau \in [0,1]$, and a holonomic section $\widetilde{F} : \mathcal{O}p\, h^1(A) \to \mathcal{R}$ such that*

$$\operatorname{dist}(\widetilde{F}(v)\,,\, F|_{\mathcal{O}p\, h^1(A)}(v)) < \varepsilon$$

for all $v \in \mathcal{O}p\, h^1(A)$.

The proof of this theorem literally repeats the proof of the original Holonomic Approximation Theorem 3.1.1. When working over a cube, the local integrability provides the first step of the induction. Then the microflexibility implies the version of the Interpolation Property 3.5.1 which is required for the proof of the Inductive Lemma 3.4.1. Finally we proceed inductively over the skeleton of the polyhedron A. Note that the homotopy extension property 13.2.2 allows us to extend the holonomic solutions obtained at each step of the induction as formal solutions to $\mathcal{O}p\, A$.

The *relative* and the *parametric* versions of Theorem 3.1.1 also hold. In the relative version the section F is assumed to be already holonomic over $\mathcal{O}p\, B$ where B is a subpolyhedron of A. The diffeomorphism h and the section $\widetilde{F}$ can be constructed in this case so that h is fixed on $\mathcal{O}p\, B$ and $\widetilde{F}$ coincides with F on $\mathcal{O}p\, B$.

13.5. Local h-principle for microflexible Diff V-invariant relations

Using Theorem 13.4.1 instead of 3.1.1 we obtain, as in Section 7.2 above the following generalization of Theorem 7.2.1

13.5.1. (Local h-principle) *All forms of the local h-principle hold for locally integrable and microflexible* Diff V*-invariant differential relations near any polyhedron $A \subset V$ of positive codimension.*

The proof repeats almost literally the proof of 7.2.1. The only problem is the construction of a homotopy between formal and genuine solutions which lies in $\mathcal{R}$. The linear homotopy does not necessarily lie in $\mathcal{R}$. In general, the construction of a homotopy in $\mathcal{R}$ requires some additional work. However, if $\mathcal{R}$ is a local neighborhood retract then one can just compress the linear homotopy into $\mathcal{R}$ by the retraction. This case is sufficient for all our further application. We leave the general case to the reader as an exercise; the key to the construction of homotopy is the first **Exercise** in 13.1.

According to 7.2.2, Theorem 13.5.1 implies

13.5.2. (Gromov [Gr69]) *Let V be an open manifold and $X \to V$ a natural fiber bundle. Then any locally integrable and microflexible* Diff$V$*-invariant differential relation $\mathcal{R} \subset X^{(r)}$ satisfies the parametric h-principle.*

When A is a manifold, and $V = A \times \mathbb{R}$ then to prove the local h-principle near $A = A \times 0 \subset V$ one does not need $\mathcal{R}$ to be invariant with respect to the whole group Diff V, but only with respect to diffeomorphisms of the form

$$(x,t) \mapsto (x, h(x,t))\,, \quad x \in V\,,\ t \in \mathbb{R}\,.$$

Indeed, in this case we need to use only this kind of diffeomorphisms in the Holonomic $\mathcal{R}$-Approximation Theorem 13.4.1 (cp. the similar Theorem 8.3.1 for open differential relations over $A \times \mathbb{R}$). The following proposition is a version of the *Main Flexibility Theorem* from [**Gr86**], p.78.

13.5.3. *Let $X \to V = A \times \mathbb{R}$ be a natural fibration and $\mathcal{R} \subset X^{(r)}$ a locally integrable and microflexible differential relation which is invariant with respect to diffeomorphisms of the form*

$$(x,t) \mapsto (x, h(x,t))\,, \quad x \in V\,,\ t \in \mathbb{R}\,.$$

Then $\mathcal{R}$ satisfies all forms of the local h-principle near $A = A \times 0$, and satisfies the parametric h-principle globally on V.

In Chapter 15 we consider more examples when the h-principle holds for differential relations invariant only with respect to a certain subgroup of the group Diff V.

Chapter 14

First Applications of Microflexibility

14.1. Subcritical isotropic immersions

A. Immersions into contact manifolds

As an immediate application of Theorem 13.5.1 and Theorem 13.5.2 we get all the forms of the local h-principle for Legendrian/isotropic immersions $\mathcal{O}p\, A \to (W, \xi)$ where, as usual, $A \subset V$ is a polyhedron of positive codimension and the parametric h-principle for Legendrian/isotropic immersions $V \to (W, \xi)$ of *open* manifolds V. Any formal/genuine *subcritical* isotropic immersion $V \to (W, \xi)$ can be extended, at least locally, to a formal/genuine isotropic immersion $V \times \mathbb{R} \to (W, \xi)$, and hence the microextension trick yields as usual all the forms of the h-principle for subcritical isotropic immersions of *closed* manifolds. This is not, however, too exciting. First, we already proved in Chapter 12 the h-principle for subcritical isotropic *embeddings*, which is much stronger than for *immersions*. Second, we will see in Chapter 16.1 that the h-principle holds even for *Legendrian* immersions (see Theorem 16.1.3) of closed manifolds.

B. Immersions into symplectic manifolds

The local h-principle for Lagrangian/isotropic immersions $\mathcal{O}p\, A \to (W, \omega)$ and the parametric h-principle for Lagrangian/isotropic immersions $V \to (W, \omega)$ of *open* manifolds V become true if we incorporate the global algebraic condition $[f^*\omega] = 0$ in the definition of a formal isotropic immersion. The proof follows from 13.5.1 and 13.5.2, though not immediately because we need to overcome the lack of microflexibility for $k = 1$ for isotropic immersions into symplectic manifolds. The microextension trick yields all the forms of the h-principle for *subcritical* isotropic immersions of *closed* manifolds. As in the case of isotropic immersions into contact manifolds, all these results are not too exciting. The only interesting thing is how one can fight the lack of microflexibility. We will explain it later in Section 16.3 where we prove the h-principle for Lagrangian immersions of closed manifolds.

14.2. Maps transversal to a contact structure

Theorem 8.3.3 which we proved earlier asserts that the parametric h-principle holds for mappings transversal to a tangent distribution, provided that the sum of the dimension of the manifold and the dimension of the distribution is *less* than the dimension of the target manifold. In his book [**Gr86**] Gromov formulated a series of exercises which culminate in a theorem which claims that if the distribution in question is *completely non-integrable* then the h-principle holds even when the sum of dimensions of the manifold and the distribution is *greater than or equal to* the dimension of the target manifold. Here the *complete non-integrability* means that the successive Lie brackets of vector fields tangent to distribution span the tangent space TW. Following Gromov's scheme we consider below the special case of maps transversal to a contact structure.

14.2.1. (Gromov [Gr86]) *Let (W, ξ) be a contact manifold. Then the maps $f : V \to W$ transversal to ξ (i.e. the maps for which $TV \xrightarrow{df} TW \to TW/\xi$ is a fiberwise surjective homomorphism) satisfy all forms of the h-principle.*

Proof. Denote by $\mathcal{R}_{\text{trans}}$ the differential relation in $J^1(V, W)$ which defines the maps $V \to (W, \xi)$ transversal to ξ. We will prove the h-principle for $\mathcal{R}_{\text{trans}}$ using a certain form of the microextension trick.

Let us recall that the differential relation $\mathcal{R}_{\text{tang}} \subset J^1(\mathbb{R}, W)$ which defines the isotropic immersions $\mathbb{R} \to (W, \xi)$, i.e. maps which are *tangent* to ξ, is locally integrable and microflexible. (see 13.1 and 13.3). Consider a mixed differential relation $\mathcal{R}_{\text{trans}-\text{tang}} \subset J^1(V \times \mathbb{R}, W)$ which corresponds to maps $V \times \mathbb{R} \to (W, \xi)$ transversal to ξ but tangent to ξ along each fiber $v \times \mathbb{R}$, $v \in V$. The openness of the relation $\mathcal{R}_{\text{trans}}$ and the local integrability and microflexibility of the relation $\mathcal{R}_{\text{tang}}$ imply the local integrability and microflexibility of the relation $\mathcal{R}_{\text{trans}-\text{tang}}$. The relation $\mathcal{R}_{\text{trans}-\text{tang}}$ is invariant with respect to diffeomorphisms of the form $(x, t) \mapsto (x, h(x, t))$, $x \in V, t \in \mathbb{R}$, and hence according to Proposition 13.5.3 all forms of the local h-principle hold for $\mathcal{R}_{\text{trans}-\text{tang}}$. The local h-principle for $\mathcal{R}_{\text{trans}-\text{tang}}$ implies the h-principle for $\mathcal{R}_{\text{trans}}$. Indeed, the restriction to $V \times 0$ of any solution of $\mathcal{R}_{\text{trans}-\text{tang}}$ on $\mathcal{O}p\,(V \times 0) \subset V \times \mathbb{R}$ is a solution of

$\mathcal{R}_{\mathrm{trans}}$ on V. On the other hand, any formal/genuine solution of $\mathcal{R}_{\mathrm{trans}}$ over any simplex Δ in V can be extended to a formal/genuine local solution of $\mathcal{R}_{\mathrm{trans-tang}}$ on $\Delta \times \mathbb{R}$. □

Similarly we can prove

14.2.2. (Immersions transversal to contact structure) *Let* (W, ξ) *be a contact manifold and* $\dim V < \dim W$. *Then the immersions* $f : V \to W$ *transversal to* ξ *satisfy all forms of the h-principle.*

Proof. Set $\mathcal{R}_{\mathrm{imm-trans}} = \mathcal{R}_{\mathrm{imm}} \cap \mathcal{R}_{\mathrm{trans}}$. Consider a mixed differential relation $\mathcal{R}_{\mathrm{imm-trans-tang}} \subset J^1(V \times \mathbb{R}, W)$ which corresponds to *immersions* $V \times \mathbb{R} \to (W, \xi)$ transversal to ξ but tangent to ξ along each fiber $v \times \mathbb{R}$, $v \in V$. The openness of the relation $\mathcal{R}_{\mathrm{imm-trans}}$ and the local integrability and microflexibility of the relation $\mathcal{R}_{\mathrm{tang}}$ imply the local integrability and microflexibility of the relation $\mathcal{R}_{\mathrm{imm-trans-tang}}$. The relation $\mathcal{R}_{\mathrm{imm-trans-tang}}$ is invariant with respect to diffeomorphisms of the form $(x, t) \mapsto (x, h(x, t))$, $x \in V, t \in \mathbb{R}$, and hence according to Proposition 13.5.3 all forms of the local h-principle hold for $\mathcal{R}_{\mathrm{imm-trans-tang}}$. The local h-principle for $\mathcal{R}_{\mathrm{imm-trans-tang}}$ implies the h-principle for $\mathcal{R}_{\mathrm{imm-trans}}$. Indeed, the restriction to $V \times 0$ of any solution of $\mathcal{R}_{\mathrm{imm-trans-tang}}$ on $\mathcal{O}p\,(V \times 0) \subset V \times \mathbb{R}$ is a solution of $\mathcal{R}_{\mathrm{imm-trans}}$ on V. On the other hand, any formal/genuine solution of $\mathcal{R}_{\mathrm{imm-trans}}$ over any simplex Δ in V can be extended to a formal/genuine local solution of $\mathcal{R}_{\mathrm{imm-trans-tang}}$ on $\Delta \times \mathbb{R}$. □

◀ **Remark.** Theorems 14.2.1 and 14.2.2 remain true (with the same proofs) for *any* distribution ξ on W for which the differential relation $\mathcal{R}_{\mathrm{tang}} \subset J^1(\mathbb{R}, W)$ which defines isotropic immersions $\mathbb{R} \to (W, \xi)$, i.e. the maps which are *tangent* to ξ, is microflexible. Thus the problem with Gromov's exercise for a general completely non-integrable distribution is how to establish the microflexibility property for the relation $\mathcal{R}_{\mathrm{tang}}$. In fact, the relation $\mathcal{R}_{\mathrm{tang}}$ is not always microflexible. As was shown by R. Bryant and L. Hsu (see [**BH93**]) most non-integrable distributions (e.g. Engel structures, which are maximally non-integrable 2-plane fields on 4-dimensional manifolds) possess rigid integral curves. In other words, the corresponding space of integral curves contains isolated points. This, of course, contradicts the microflexibility of the corresponding relation $\mathcal{R}_{\mathrm{tang}}$. However, it is possible that the space of curves which violate the microflexibility always has infinite codimension in the space of (local) solutions of $\mathcal{R}_{\mathrm{tang}}$. This would be sufficient in order to extend Theorems 14.2.1 and 14.2.2 to general completely non-integrable distributions. We leave to the reader the pleasure of finding and proving an appropriate microflexibility result for $\mathcal{R}_{\mathrm{tang}}$, and thus completing the solution of Gromov's exercise in the general case. ▶

Theorem 14.2.2 together with the h-principle for contact structures on open manifolds (see 10.3.2) implies the following theorem of McDuff about maximally non-integrable tangent hyperplane distributions on even-dimensional manifold. This theorem is a contact analog of Theorem 10.4.1.

Let ξ be a tangent hyperplane field on a $2n$-dimensional manifold W. We say that ξ is *maximally non-integrable* if the form $d\alpha|_\xi$ has the maximal rank $2n-2$, where α is a defining 1-form for ξ (which is valued in a non-trivial line bundle TW/ξ if ξ is not coorientable). For simplicity we formulate the h-principle below only for the case of coorientable distributions, leaving the general case as an exercise to the reader.

14.2.3. (McDuff [MD87a]) *Let $\xi = \operatorname{Ker}\alpha$ be a hyperplane field on a $2n$-dimensional manifold V and ω a 2-form whose restriction to ξ is of maximal rank $2n-2$. Then V admits a maximally non-integrable hyperplane distribution $\widetilde{\xi} = \operatorname{Ker}\widetilde{\alpha}$ such that (ξ,ω) and $(\widetilde{\xi}, d\widetilde{\alpha})$ are homotopic in the space of pairs (tangent hyperplane distribution η, 2-form of maximal rank on η).*

Proof. Let $\widehat{\xi} \subset T\widehat{V} = T(V\times\mathbb{R})$ be the Whitney sum of ξ, pulled back to $\widehat{V}$, and the trivial line bundle tangent to the second factor. The form ω on V extends in a homotopically canonical way to a 2-form $\widehat{\omega}$ on $\widehat{V}$ such that its restriction to $\widehat{\xi}$ is non-degenerate. Hence, Gromov's h-principle for contact structures on open manifolds provides a contact structure $\overline{\xi}$ on $\widehat{V}$ in the formal homotopy class prescribed by the pair $(\widehat{\xi},\widehat{\omega})$. Unfortunately, $\overline{\xi}$ is not necessarily transversal to $V = V\times 0 \subset \widehat{V}$. However, by applying Theorem 14.2.2 we can deform V via a *regular homotopy* to achieve transversality to $\overline{\xi}$. Then the distribution $\widetilde{\xi}$ induced on V by this transversal map from $\overline{\xi}$ has the required properties. □

A similar argument proves also the parametric and relative versions of the h-principle 14.2.3.

Chapter 15

Microflexible $\mathfrak{A}$-invariant Differential Relations

We generalize in this section the local h-principle 13.5.1 to a class of differential relations which are invariant only with respect to a certain subgroup of Diff V. Let the abstract definition of the allowable class of subgroups do not mislead the reader: the only interesting applications which we consider below are concerned with the groups of symplectic and contact diffeomorphisms.

15.1. $\mathfrak{A}$-invariant differential relations

Let $X \to V$ be a natural fibration. Given a subgroup $\mathfrak{A} \subset \operatorname{Diff} V$, the relation $\mathcal{R}$ is called $\mathfrak{A}$-invariant if $h_*(\mathcal{R}) = \mathcal{R}$ for all $h \in \mathfrak{A}$. For instance, the relations $\mathcal{R}_{\text{isosymp}}$ and $\mathcal{R}_{\text{isocont}}$ which defines isosymplectic and isocontact immersions are not Diff V-invariant. However, they are invariant with respect to the subgroups of symplectic and contact diffeomorphisms, respectively.

Let $\mathfrak{A}$ be a Lie subgroup of the group of compactly supported diffeomorphisms of V, and $\mathfrak{a}$ its Lie algebra of vector fields. We call $\mathfrak{a}$ (and $\mathfrak{A}$) *capacious* if it satisfies the following two conditions:

(CAP_1) for any $v \in \mathfrak{a}$, any compact subset $A \subset V$ and its neighborhood $U \supset A$ there exists a vector field $\tilde{v}_{A,U} \in \mathfrak{a}$ which is supported in U and which coincides with v on A;

(CAP$_2$) given any tangent hyperplane $\tau \subset T_x(V)$, $x \in V$, there exists a vector field $v \in \mathfrak{a}$ which is transversal to τ.

Moreover, we require both properties CAP$_1$ and CAP$_2$ to hold parametrically for any compact space of parameters.

The two examples of capacious subgroups most important for us are the identity component of the group of compactly supported contact diffeomorphisms of a contact manifold, and the group of compactly supported Hamiltonian diffeomorphisms of a symplectic manifold.

◀ **Remark.** The notion of a capacious subgroup of diffeomorphisms is a version of the notion of a *set of sharply moving diffeotopies* in Gromov's book [**Gr86**]. ▶

15.2. Local h-principle for microflexible $\mathfrak{A}$-invariant relations

15.2.1. (Local h-principle) *Let $\mathfrak{A} \subset \mathrm{Diff}\, V$ be a capacious subgroup, $X \to V$ a natural fibration, and $\mathcal{R}$ an $\mathfrak{A}$-invariant locally integrable and microflexible differential relation. Then all forms of the local h-principle hold for $\mathcal{R}$ near any subpolyhedron $A \subset V$ of positive codimension. In particular, if V is a symplectic (contact) manifold then the local h-principle holds for* $\mathrm{Ham}\, V$*-invariant (*$\mathrm{Diff}_{\mathrm{cont}} V$*-invariant) locally integrable and microflexible differential relations.*

◀ **Remark.** As we will see below, only the invariance of the differential relation $\mathcal{R}$ with respect to arbitrarily C^0-small diffeomorphisms from the capacious group $\mathfrak{A}$ will be used in the proof. Hence, *the h-principle 15.2.1 remains true if $\mathcal{R}$ is invariant only with respect to diffeomorphisms from an arbitrarily small C^0-neighborhood of* Id *in the group $\mathfrak{A}$.* ▶

Proof. The only problem here, compared with the proof of the h-principle 13.5.1 for microflexible Diff$V$-invariant relations, is that the fibered shifting diffeotopy h^τ, provided by the Holonomic $\mathcal{R}$-approximation Theorem 13.4.1, does not necessarily belong to the subgroup $\mathfrak{A}$. Let us recall here the scheme of the proof of 13.4.1 and show how this problem could be corrected.

First we observe that the property CAP$_2$ guarantees that the polyhedron A can be subdivided to ensure that each of its simplices Δ admits a transversal vector field $v_\Delta \in \mathfrak{a}$. Next we choose near each simplex Δ a coordinate system as in Lemma 3.2.1, which identifies a slightly smaller domain in the simplex with the coordinate cube I^k, and the vector field v_Δ with the coordinate vector field $\frac{\partial}{\partial x_n}$. The key ingredient in the proof of Lemma 3.2.1 is the Inductive Lemma 3.4.1 and its second version 3.4.2. No changes are necessary in the proof of the Inductive Lemma 3.4.1 (except that we substitute the openness condition by local integrability and microflexibility of $\mathcal{R}$). As a result we obtain a family of holonomic sections $\widetilde{F}_z : \Omega_z \to \mathcal{R}$ defined on domains

$$\Omega_z = \bigcup_{i=1}^{N} (\widetilde{U}_{i,z} \setminus A_{i,z}) \cup \bigcup_{1}^{N} \mathcal{O}p\, B_{i,z}\,,$$

where we use the notation introduced in Lemmas 3.5.1 and 3.5.2, i.e.

$$B_{i,z} = z \times i\sigma \times I^l\,, i = 0, \dots, N,$$

and for $i = 1, \dots, N$

$$\widetilde{U}_{i,z} = \mathcal{N}_\delta((z \times c_i \times I^l) \cap \{c_i - \sigma/2 < t < c_i + \sigma/2\}\,,$$

$$A_{i,z} = \left(\overline{U_{\delta/4}(z \times c_i \times I^l)} \setminus V_\delta(z, c_i)\right) \cap \{(x_1, \dots, x_{k-l-1}) = z,\ x_{k-l} = c_i\},$$

where $c_i = \frac{2i-1}{2N}$ and $\sigma = \frac{1}{N}$.

We also set

$$\widetilde{U}_i = \bigcup_{z \in I^{k-l-1}} \widetilde{U}_{i,z}\,,\ \ A_i = \bigcup_{z \in I^{k-l-1}} A_{i,z}\,,\ \ B_i = \bigcup_{z \in I^{k-l-1}} B_{i,z}\,,\ \ i = 1, \dots, N\,.$$

To finish the proof we need the following analog of 3.4.2

15.2.2. *For $i = 1, \dots, N$ there exist compactly supported diffeotopies $h_i^\tau : \widetilde{U}_i \to \widetilde{U}_i$, $\tau \in [0,1]$, such that*

- $h_i^\tau \in \mathfrak{A}$;
- $h_i^1(I^k \cap \widetilde{U}_i) \cap A_i = \varnothing$.

Proof. of 15.2.2 By construction the vector field $v = \frac{\partial}{\partial x_n}$ belongs to $\mathfrak{a}$. Using the property CAP_1 for each $i = 1, \dots, N$ we can find a field $\tilde{v}_i \in \mathfrak{A}$ which coincides with v_i on

$$\mathcal{N}_{\delta/2}(A_i) \cap \{c_i - \sigma/4 \le t \le c_i + \sigma/4\}$$

and is supported in a slightly bigger subset of $\widetilde{U}_i$. Let $e^{\tau \tilde{v}_i}$ be the flow generated by the vector field $\tilde{v}_i$. Let l be any line parallel to the x_n-axis which intersects the set A_i. Set

$$\lambda = A_i \cap l = \{-\delta/4 \le x_n \le \delta/4\} \cap l$$

and let $\bar{\lambda}$ be the interval

$$\{-3\delta/4 \le x_n \le -\delta/4\} \cap l\,.$$

The flow $e^{\tau\tilde{v}_i}$ for the time $T = \delta/2$ slides $\bar{\lambda}$ along l with speed 1 and hence $e^{T\tilde{v}_i}(\bar{\lambda}) = \lambda$. Therefore the isotopy h_i^τ on $\widetilde{U}_i$ which is defined by rescaling the time parameter of this flow:

$$h_i^\tau = e^{(\delta/2)\tau\tilde{v}_i}\,, \quad \tau \in [0,1]\,,$$

disjoins I^k from A_i. □

Now we can complete the proof of 15.2.1. Notice that the isotopies h_i^τ, $i = 1, \dots, N$, fit together into a smooth isotopy $h^\tau \in \mathfrak{A}$ which is defined on $\mathcal{O}p\, I^k$, and has the property that the image $h^1(z \times I \times I^l)$ is contained in the domain Ω_z where the holonomic section $\widetilde{F}_z$ provided by the Inductive Lemma 3.4.1 is defined. Hence, one can use h^1 to pullback the section $\widetilde{F}_z$ to a neighborhood of $z \times I \times I^l$ and thus continue inductively in constructing the solution of $\mathcal{R}$ precisely as in the proof of Theorem 7.2.1. □

Chapter 16

Further Applications to Symplectic Geometry

16.1. Legendrian and isocontact immersions

As an application of the h-principle 15.2.1 we get

16.1.1. (Local h-principle for isocontact immersions) *Let (V, ξ_V) and (W, ξ_W) be contact manifolds and $A \subset V$ a polyhedron of positive codimension. Then all forms of the local h-principle hold for isocontact immersions $(\mathcal{O}p\, A, \xi_V|_{\mathcal{O}p\, A}) \to (W, \xi_W)$.*

Indeed, the corresponding differential relation $\mathcal{R}_{\text{isocont}}$ is locally integrable, microflexible, and invariant with respect to the capacious group $\text{Diff}_{\text{cont}}(V)$.

Theorem 16.1.1 and the microextension trick imply the following

16.1.2. (Homotopy principle for isocontact immersions, Gromov [Gr86]) *All forms of the h-principle hold for isocontact immersions $(V, \xi_V) \to (W, \xi_W)$ as long as $\dim V < \dim W$.*

Proof. Let N be the normal bundle to $F(\xi_V) \in \xi_W$ with respect to the conformal symplectic structure $\text{CS}(\xi_W)$. Then N has the structure of a symplectic vector bundle. According to Lemma 9.4.1 there exists a contact structure ξ_N on a neighborhood $\mathcal{O}p\, V$ of the 0-section in the total space of the bundle N such that (V, ξ_V) is a contact submanifold of $(\mathcal{O}p\, A, \xi_N)$, the fibers of the bundle N are tangent to $\xi_N|_V$ and serve as orthogonal complements of ξ_V in ξ_N with respect to the conformal symplectic structure $\text{CS}(\xi_N)$. Then any isocontact homomorphism $F : (TV, \xi_V) \to (TW, \xi_W)$ canonically extends to an *equidimensional* isocontact homomorphism $\widehat{F}$:

$(\mathcal{O}p\,V, \xi_N) \to (TW, \xi_W)$. On the other hand, any isocontact immersion $(\mathcal{O}p\,V, \xi_N) \to (W, \xi_W)$ restricts to V as an isocontact immersion $(V, \xi_V) \to (W, \xi_W)$. □

Similarly, any formal Legendrian immersion $F : TV \to TW$ can be canonically extended to a formal *equidimensional* isocontact immersion

$$\widehat{F} : T(J^1(V, \mathbb{R})) \to TW\,.$$

Hence, we get

16.1.3. (Homotopy principle for Legendrian immersions; Gromov [Gr71], Duchamp [Du84]) *All forms of h-principle hold for Legendrian immersions $V \to (W, \xi)$.*

Of course, the h-principle for *subcritical* isotropic immersions into contact manifolds also follows from 16.1.2. However, as we already noted in Chapter 13.5 this h-principle also follows from Theorem 13.5.2 which we proved earlier.

◀ **Remark.** Parametric forms of all the h-principles considered in this section remain true (with the same proof) in the fibered form (see 6.2 E), i.e. when the contact structures on the source, target, or both are also allowed to vary with the parameter. ▶

16.2. Generalized isocontact immersions

We will now generalize Theorem 16.1.2 for isometric mappings of arbitrary, not necessarily contact distributions. Let η be an arbitrary tangent distribution of codimension s on a manifold V and ξ a contact structure on a manifold W. An immersion $f : (V, \eta) \to (W, \xi)$ is called *isocontact* if it is transversal to ξ and $df(\eta) \subset \xi$. Let us now formulate the corresponding formal notion. Let $\eta^* \subset T^*V$ be the bundle conormal to η. Sections of η^* are 1-forms annihilating the distribution η. Let $D\eta$ be the vector bundle which is pointwise generated by the sections $d\theta|_\eta$, where $\theta \in \operatorname{Sec}\eta^*$. Note that any section of $D\eta$ has the form $d\theta|_\eta$, where $\theta \in \operatorname{Sec}\eta^*$. Indeed, for any forms $\theta_i \in \operatorname{Sec}\eta^*$ and functions $f_i : V \to \mathbb{R}$, $i = 1, \dots, k$, we have

$$\sum_1^k f_i(d\theta_i|_\eta) = \big(d\sum_1^k f_i d\theta_i\big)\big|_\eta = d\theta|_\eta\,,$$

where $\theta = \sum\limits_1^k f_i d\theta_i \in \operatorname{Sec}\theta^*$. If $g : (V, \eta) \to (W, \xi)$ is an isocontact immersion and $\xi = \operatorname{Ker}\alpha$ then $\alpha \circ dg \in \operatorname{Sec}\eta^*$, and hence $(g^* d\alpha)|_\eta$ is a section of the bundle $D\eta$. This motivates the following definition:

A monomorphism $F : (TV, \eta) \to (TW, \xi = \operatorname{Ker} \alpha)$ is called *isocontact* if it is transversal to ξ and $\big(F^* d\alpha\big)\big|_\eta \in \operatorname{Sec} D\eta$.

16.2.1. (Gromov [Gr86] and Datta [Da97]) *Let (W, ξ) be a contact manifold and η an arbitrary tangent distribution on a manifold V. Let $A \subset V$ be a polyhedron of positive codimension.*

- **(a)** *All forms of the local h-principle hold for isocontact immersions $(\mathcal{O}p\, A, \eta) \to (W, \xi)$.*
- **(b)** *If $\dim W > \dim V$, then all forms of the global h-principle hold for isocontact immersions $(V, \eta) \to (W, \xi)$.*

To prove 16.2.1 we will need the following lemma

16.2.2. *Let $\eta = \operatorname{Ker} \alpha$ be a tangent hyperplane distribution on a manifold V and $\pi : E \to V$ a vector bundle over V. Let us denote by $\widehat{\eta}$ the subbundle of $T(E)|_V$ which is the direct sum of η and the vector bundle $E \to V$ viewed as subbundle of the bundle $T(E)|_V$. Suppose that there exists a not necessarily closed 2-form Ω on E such that $\Omega|_{\widehat{\eta}}$ is non-degenerate and $\Omega|_\eta = d\alpha|_\eta$. Then $\widehat{\eta}$ extends to a contact structure $\overline{\eta}$ on E.*

Proof. of 16.2.2 Using an appropriate partition of unity we can reduce the proof to the case when the bundle $E \to V$ is trivial, $E = W \times \mathbb{R}^s$. Let $t_1, \dots, t_s$ be the coordinates corresponding to the second factor. The form Ω can be presented as the sum $\Omega = \Omega' + d\widetilde{\alpha}$ where $\widetilde{\alpha} = \pi^* \alpha$, such that $\Omega'|_\tau = 0$. In particular, the form Ω' can be written as

$$\Omega' = \widetilde{\alpha} \wedge \beta + \sum_1^k dt_i \wedge \beta_i \,.$$

Consider the 1-form

$$\overline{\alpha} = \widetilde{\alpha} + \sum_1^s t_i \alpha_i \,.$$

The distribution $\overline{\eta} = \operatorname{Ker} \overline{\alpha}$ coincides with $\widehat{\eta}$ along V, and $d\overline{\alpha}_{\widehat{\eta}} = \Omega|_\eta$. In particular, $\overline{\eta}$ is a contact structure on $\mathcal{O}p\, V \subset E$. ☐

Proof. of 16.2.1 The global h-principle in the case $\dim W > \dim V$ follows from the local one via the standard microextension argument. So we will prove here only the local h-principle and only in the non-parametric case, and leave the general case as an exercise to the (already very experienced) reader.

Let $F : (T(\mathcal{O}p\, A), \eta) \to (W, \xi)$ be an isocontact homomorphism. Let $\xi = \operatorname{Ker} \alpha$. Set $\beta = f^* \alpha$ and $\Omega = F^* d\alpha$. Set $\tau = \operatorname{Ker} \beta$ and let $E \to \mathcal{O}p\, A$ be the vector bundle whose fiber over a point $v \in \mathcal{O}p\, A$ is the orthogonal complement (with respect to some Riemannian metric) to $F(\tau_v)$ in $\xi_{f(v)}$, where $f : \mathcal{O}p\,A \to W$ is the map underlying the homomorphism F. Consider the subbundle of $\widehat{\tau} \subset T(E)|_{\mathcal{O}p\,A}$ which is spanned by τ and the fibers of the vector bundle E. The subbundle $\widehat{\tau}$ is symplectic. The linear symplectic structure on its fibers is given by the pull-back Ω of the symplectic form $d\alpha$ on ξ under the isomorphism $\widehat{F} : \widehat{\tau} \to \xi$ which canonically extends the injection $F : \tau \to \xi$. Hence, we can first use Lemma 16.2.2 to extend $\widehat{\tau}$ to a contact structure $\overline{\tau}$ on a neighborhood $\mathcal{O}p_E A$ of the 0-section $\mathcal{O}p\,A$ in E, and then apply the local h-principle 16.1.1 for isocontact immersions $g : (\mathcal{O}p_E A, \overline{\tau}) \to (W, \xi)$. Then the restriction of this map to a neighborhood $\mathcal{O}p\,A$ of A in V is automatically an isocontact immersion $(\mathcal{O}p\,A, \tau) \to (W, \xi)$. Combined with the inclusion $\eta \hookrightarrow \tau$ it gives us the required isocontact immersion $(\mathcal{O}p\,A, \eta) \to (W, \xi)$. □

16.3. Lagrangian immersions

Let us recall that a symplectic manifold (W, ω) is called *exact* if $\omega = d\alpha$. A Lagrangian immersion $f : V \to (W, \omega = d\alpha)$ is called *exact* if the closed form $f^*\alpha$ is exact.

16.3.1. *Let $(W, \omega = d\alpha)$ be an exact $2n$-dimensional symplectic manifold. Then for any n-dimensional manifold V all forms of the h-principle holds for exact Lagrangian immersions $V \to W$. In particular, any family of isotropic monomorphisms $TV \to (TW, \omega)$ is homotopic in this class to a family of differentials of exact Lagrangian immersions. Moreover, the same is true when the symplectic form ω on W depends itself on the parameter.*

Proof. Any isotropic monomorphism $F : TV \to TW$ homotopically canonically lifts to an isotropic monomorphism into the contact bundle

$$\xi = \mathrm{Ker}\,(dz - \alpha)$$

on $W \times \mathbb{R}$. Conversely, any Legendrian immersion $V \to (W \times \mathbb{R}, \xi)$ projects to an *exact* Lagrangian immersion into $(W, d\alpha)$. Thus it remains to apply Theorem 16.1.3 (and the remark which follows it). □

16.3.2. (Homotopy principle for Lagrangian immersions; Gromov [Gr71], Lees [Le76]) *For any n-manifold V and symplectic $2n$-manifold (W, ω) all forms of the h-principle hold for Lagrangian immersions $V \to (W, \omega)$ as long as the cohomological condition $[f^*\omega] = 0$ is incorporated in the definition of formal solutions of $\mathcal{R}_{\mathrm{lag}}$.*

Proof. Let $F : TV \to TW$ be an isotropic monomorphism such that $[f^*\omega] = 0$. Using Smale-Hirsch's h-principle for immersions (see 8.2.1) we may assume from the very beginning that $f = \mathrm{bs}\, F$ is a differentiable immersion. Let N be the total space of the normal bundle to $f(V)$ in W. The immersions $f : V \to W$ can be extended to immersions $\widetilde{f} : N \to W$, so that the $F = d\widetilde{f} \circ G$. The monomorphism $G : TV \to TN$ is isotropic with respect to the induced symplectic form $\widetilde{\omega} = \widetilde{f}^*\omega$. By assumption this form is exact, $\widetilde{\omega} = d\alpha$, and hence one can use 16.3.1 to finish the proof in the non-parametric case.

Now let $F_t : TV \to TW,\ t \in D^k$, be any *family* of isotropic monomorphisms such that $[f_t^*\omega] = 0$, and $F_t = df_t$ for $t \in \partial D^k$, where f_t is the map $V \to W$ underlying F_t. Using Smale-Hirsch's h-principle for immersions we can deform the family F_t into a family of immersions. Moreover, the family F_t could be kept unperturbed for $t \in \partial D^k$. Let N be the total space of the normal bundle to $f_0(V)$ in W. The immersions $f_t : V \to W$ can be extended to immersions $\widetilde{f}_t : N \to W$ so that the family F_t can be decomposed as $F_t = d\widetilde{f}_t \circ G_t$, $t \in D^k$. The monomorphism $G_t : TV \to TN$ is isotropic with respect to the induced symplectic form $\widetilde{\omega}_t = \widetilde{f}_t^*\omega$, $t \in D^k$. By assumption these forms are exact, $\omega_t = d\alpha_t$. Unfortunately, now we cannot use 16.3.1 because our original Lagrangian immersions $f_t : V \to (W, \omega)$, $t \in \partial D^k$ may well be non-exact. This is, however, a minor problem because instead of α_t one can take a new family of primitives $\{\alpha_t - \widetilde{\alpha}_t\}_{t \in D^k}$ for $\widetilde{\omega}_t$, where $\widetilde{\alpha}_t$ is a family of *closed* 1-forms on N such that

$$\widetilde{\alpha}_t = p^*(\alpha_t|_V) \ \text{ for } \ t \in \partial D^k$$

where $p : N \to V$ is the projection. With respect to this new family of primitives the Lagrangian immersions $f_t : V \to (W, \omega)$, $t \in \partial D^k$ are exact and we can apply 16.3.1 to finish the proof. ☐

16.4. Isosymplectic immersions

Let now (V, ω_V) and (W, ω_W) be two symplectic manifolds, $\dim W \geq \dim V$, and A a subpolyhedron of positive codimension in V.

16.4.1. *Let U be an open subset of the product $V \times W$ such that the form $\Omega = \omega_V \oplus \omega_W$ is exact on U, $\Omega|_U = d\alpha$. Then all forms of the local h-principle hold for Lagrangian (isotropic) sections $s : \mathcal{O}p\, A \to U \subset V \times W$.*

Proof. As in the proofs of 16.3.1 and 16.3.2 above we can reduce the problem for Lagrangian (isotropic) sections $\mathcal{O}p\, A \to U$ to the problem about Legendrian (isotropic) sections $\mathcal{O}p\, A \to (U \times \mathbb{R}, \xi = \mathrm{Ker}\,(dz - \alpha))$. The differential relation which corresponds to the Legendrian problem is locally integrable, microflexible and invariant with respect to a sufficiently small neighborhood of the identity in the capacious group $\mathrm{Ham}(V,\omega)$. Hence, according to Theorem 15.2.1 we deduce the required local h-principle. $\square$

Let $\mathcal{R}_{\mathrm{isosymp}}$ be the differential relation corresponding to the problem of isosymplectic immersions $(V,\omega_V) \to (W,\omega_W)$. For a polyhedron $A \subset V$ of positive codimension we denote by $\mathrm{Sec}^0_{\mathcal{O}p\,A}\,\mathcal{R}_{\mathrm{isosymp}}$ the subspace in $\mathrm{Sec}_{\mathcal{O}p\,A}\,\mathcal{R}_{\mathrm{isosymp}}$ which consists of sections $F : \mathcal{O}p\,A \to J^1(V,W)$ which satisfy the cohomological condition

$$[f^*\omega_W] = [\omega_V|_{\mathcal{O}p\,A}]\,,$$

where $f : \mathcal{O}p\,A \to W$ is the map underlying the section F.

16.4.2. (Local h-principle for isosymplectic immersions) *All forms of the local h-principle hold for the inclusion*

$$\mathrm{Hol}_{\mathcal{O}p\,A}\,\mathcal{R}_{\mathrm{isosymp}} \to \mathrm{Sec}^0_{\mathcal{O}p\,A}\,\mathcal{R}_{\mathrm{isosymp}}$$

near any polyhedron A of positive codimension.

Proof. To spare the reader from extra indices we consider here only the non-parametric case. Let F be a section from $\mathrm{Sec}^0_{\mathcal{O}p\,A}\,\mathcal{R}_{\mathrm{isosymp}}$ and $f : \mathcal{O}p\,A \to W$ its underlying map. Let us choose an open neighborhood U of the 0-jet part of the section F, i.e. of the graph $\widehat{f}(x) = (x, f(x))$, $x \in \mathcal{O}p\,A$. Note that isosymplectic immersions $\mathcal{O}p\,A \to W$ can be characterized by the property that their graphs are isotropic with respect to the symplectic form $\Omega = \omega_V \oplus (-\omega_W)$ on $J^0(V,W) = V \times W$. By assumption the form $\Omega|_W$ is exact, and hence we can use Lemma 16.4.1 to C^0-approximate $\widehat{f}$ by an isotropic section $x \mapsto (x, g(x))$, $x \in \mathcal{O}p\,A$. Then $g : \mathcal{O}p\,A \to W$ is the required isosymplectic immersion. $\square$

Theorem 16.4.2 and the microextension trick imply the following theorem

16.4.3. (Homotopy principle for isosymplectic immersions, Gromov [Gr86]) *All forms of the h-principle hold for isosymplectic immersions $(V,\omega_V) \to (W,\omega_W)$ as long as* $\dim V < \dim W$ *and the algebraic condition $[f^*\omega_W] = [\omega_V]$ is incorporated in the definition of formal solutions of $\mathcal{R}_{\mathrm{symp-iso}}$.*

Proof. Let U be the ω_W-normal bundle to $F(TV)$ in TW. According to 9.2.2 a neighborhood $\mathcal{O}p\,V$ of the 0-section V in the total space of this bundle admits a symplectic form ω_N such that $\omega_N|_V = \omega_V$ and the fibers of the bundle N are ω_N-orthogonal to TV in $TN|_V$. Any formal isosymplectic immersion $F : (TV,\omega_V) \to (TW,\omega_W)$ then canonically extends to a formal *equidimensional* isosymplectic immersion $\widehat{F} : (T(\mathcal{O}p\,V),\omega_N) \to (TW,\omega_W)$. Hence the result follows from Theorem 16.4.2. $\square$

16.5. Generalized isosymplectic immersions

The h-principle for isosymplectic immersions can be generalized to the case when the form on the source manifold is not necessarily non-degenerate.

Let σ be a closed 2-form on a manifold V, and (W,ω) be a symplectic manifold. An immersion $f:(V,\sigma)\to(W,\omega)$ is called isosymplectic if $f^*\omega=\sigma$. A monomorphism $F:(TV,\sigma)\to(TW,\omega)$ is called isosymplectic if $F^*\omega=\sigma$ and the equality $f^*[\omega]=[\sigma]$ holds for the cohomology classes of the forms σ and ω where $f:V\to W$ is the map which underlies the homomorphism F. Let us denote by $\mathrm{Iso}(V,\sigma;W,\omega)$ and $\mathrm{iso}(V,\sigma;W,\omega)$ the space of isosymplectic immersions $(V,\sigma)\to(W,\omega)$ and the space of isosymplectic homomorphisms $(TV,\sigma)\to(TW,\omega)$, respectively. The derivation map defines a natural inclusion $D:\mathrm{Iso}(V,\sigma;W,\omega)\hookrightarrow\mathrm{iso}(V,\sigma;W,\omega)$.

16.5.1. *Let (V,σ) and (W,ω) be as above and $\dim W>\dim V$. Then the map*

$$D:\mathrm{Iso}(V,\sigma;W,\omega)\hookrightarrow\mathrm{iso}(V,\sigma;W,\omega)$$

is a homotopy equivalence.

To prove Theorem 16.5.1 we will need the following lemma, which is an analog of Lemma 16.2.2 which we proved above in the contact case.

16.5.2. *Let η be a closed 2-form on a manifold W and $\pi:E\to W$ a vector bundle over W. Suppose that there exists a non-degenerate, not necessarily closed 2-form Ω on E with $\Omega|_W=\eta$. Then there exists a symplectic form ω on $\mathcal{O}p\,W\subset E$ such that $\Omega|_{TE|_W}=\omega|_{TE|_W}$.*

Proof of 16.5.1. The form Ω can be presented as the sum $\Omega=\Omega'+\pi^*\eta$, where $\Omega'|_W=0$. Suppose first that the fibration $E\to W$ is trivial, so that $E=W\times\mathbb{R}^s$. Let $t_1,\dots,t_s$ be the coordinates corresponding to the second factor. The form Ω' can be presented in the form $\Omega'=\sum_1^k dt_i\wedge\alpha_i$. Then the closed 2-form

$$\omega=\pi^*\eta+\sum_1^k d(t_i\alpha_i)$$

coincides on $TE|_W$ with Ω, and hence it is non-degenerate in a neighborhood $\mathcal{O}p\ W\subset E$.

The case of a non-trivial fibration $E\to W$ can be reduced to the just considered case by choosing an appropriate partition of unity on W. ☐

Proof. of Theorem 16.5.1 Let $F:(TV,\sigma)\to(TW,\omega)$ be an isosymplectic immersion. Let $E\to V$ be the normal bundle to the formal immersion F, i.e. its fiber over a point $v \in V$ is the orthogonal complement to $F(T_v V)$ in $T_{f(v)}W$, where the homomorphism F covers the map $f : V \to W$. The injective homomorphism F canonically extends to an isomorphism $\widetilde{F} : TE \to TW$. The 2-form $\widetilde{\sigma} = F^*\omega$ is non-degenerate and coincides with σ on v. Applying Lemma 16.5.2 we get a symplectic form η on $\mathcal{O}p\, V$ such that $\widetilde{\sigma}|_{TN|_V} = \eta|_{TN|_V}$. Then the local h-principle 16.4.2 allows us to construct an isosymplectic embedding $g : (\mathcal{O}p\, V, \eta) \to (W, \omega)$. The restriction of g to V is the required isosymplectic immersion $(V, \sigma) \to (W, \omega)$. This finishes off the proof in the non-parametric case. The generalization to the parametric case is straightforward. □

Part 4

Convex Integration

Chapter 17

One-Dimensional Convex Integration

17.1. Example

Let us call a path

$$r : I = [0,1] \to \mathbb{R}^2,\ \ r(t) = (x(t), y(t)),$$

short if $\dot{x}^2 + \dot{y}^2 < 1$. The graph of a short path in the space-time $\mathbb{R}^3 = \mathbb{R} \times \mathbb{R}^2$ (where the space is two-dimensional) is a time-like world line: its tangent line at a point (t_0, x_0, y_0) lies inside the light cone

$$(t - t_0)^2 \geq (x - x_0)^2 + (y - y_0)^2,$$

assuming that the speed of light $c = 1$ (see Fig.17.1).

Figure 17.1. The graph of a slow path.

◀ **Exercise.** Prove that any short path can be approximated by a solution of the equation $\dot{x}^2 + \dot{y}^2 = 1$. In other words, the world line of a particle can be C^0-approximated by the world line of a photon. (Hint: any path of length l can be C^0-approximated by a path of length L for any $L > l$.) ▶

The above exercise implies that the space of solutions $I \to \mathbb{R}^2$ of the differential equation $\dot{x}^2 + \dot{y}^2 = 1$ is C^0-dense in the space of solution of differential inequality $\dot{x}^2 + \dot{y}^2 < 1$.

The exercise illuminates the following idea: given a first order differential relation for maps $I \to \mathbb{R}^q$, it is useful to consider a "relaxed" differential relation which is the fiberwise convex hull of the original relation. A direct implementation of this idea in the context of Ordinary Differential Equations is known to specialists in Control Theory as Filippov's Relaxation Theorem ([**Fi67**]). A far more subtle implementation in the context of Partial Differential Equations is known to specialists in Differential Topology as Gromov's Convex Integration Theory ([**Gr73**], [**Gr86**]). It is interesting to mention that for a long time specialists from both sides did not know about the existence of a parallel theory. It was D. Spring who pointed out the connection; see the discussion and historical remarks in [**Sp98**].

17.2. Convex hulls and ampleness

A differential relation $\mathcal{R} \subset J^1(\mathbb{R}, \mathbb{R}^q)$ can be thought of, in the spirit of Control Theory, as a *differential inclusion*

$$\dot{y} \in \Omega(t, y), \ t \in \mathbb{R}, \ y \in \mathbb{R}^q,$$

where $\Omega(t, y) = \mathcal{R} \cap P_{t,y}$ and $P_{t,y} \simeq \mathbb{R}^q$ is the fiber of the projection

$$J^1(\mathbb{R}, \mathbb{R}^q) \to J^0(\mathbb{R}, \mathbb{R}^q) = \mathbb{R} \times \mathbb{R}^q$$

over a point $(t, y) \in \mathbb{R} \times \mathbb{R}^q$. This fiber can be identified with the tangent space $T_y(t \times \mathbb{R}^q) = T_y\mathbb{R}^q$, which, by definition, consists of all vectors in $\mathbb{R}^q$ originating at the point $y \in \mathbb{R}^q$.

◀ **Remark.** We use here the letter t for the one-dimensional independent variable while reserving the letter x for the *n-dimensional* independent variable in our further generalizations in 18.2. ▶

Given an affine space P, a set $\Omega \subset P$ and a point $y \in \Omega$ we will denote by $\mathrm{Conn}_y \Omega$ a path-connected component of Ω which contains y and by $\mathrm{Conv}_y \Omega$ the convex hull of $\mathrm{Conn}_y \Omega$. A set $\Omega \subset P$ is called *ample* if $\mathrm{Conv}_y \Omega = P$ for all $y \in \Omega$. Note that according to this definition the *empty set is ample.* A differential relation $\mathcal{R} \subset J^1(\mathbb{R}, \mathbb{R}^q)$ is called *ample* if $\mathcal{R}$ is fiberwise ample, i.e. $\Omega(t, y) \subset \mathbb{R}^q$ is ample for every $(t, y) \in \mathbb{R} \times \mathbb{R}^q$.

Given a differential relation $\mathcal{R} \subset J^1(\mathbb{R}, \mathbb{R}^q)$ and a section $F = (f, \varphi) : \mathbb{R} \to \mathcal{R}$ we will use the following notation:

- $\operatorname{Conn}_{F(t)} \mathcal{R}$ - the path-connected component of $\Omega(f(t))$ which contains $F(t)$ in the fiber $P_{(t,f(t))}$ of the fibration $p_0^1 : J^1(\mathbb{R}, \mathbb{R}^q) \to J^0(\mathbb{R}, \mathbb{R}^q)$ over the point $(t, f(t))$;
- $\operatorname{Conv}_{F(t)} \mathcal{R}$ - the convex hull of $\operatorname{Conn}_{F(t)} \mathcal{R}$;
- $\operatorname{Conv}_F \mathcal{R}$ - the differential relation $\bigcup\limits_{t \in \mathbb{R}} \operatorname{Conv}_{F(t)} \mathcal{R}$ in $J^1(\mathbb{R}, \mathbb{R}^q)$.

We will call a formal solution $F = (f, \varphi)$ of $\mathcal{R} \subset J^1(\mathbb{R}, \mathbb{R}^q)$ *short*, if f is a solution of $\operatorname{Conv}_F \mathcal{R}$.

Given a *fiberwise path-connected* differential relation $\mathcal{R} \subset J^1(\mathbb{R}, \mathbb{R}^q)$, we will denote by $\operatorname{Conv} \mathcal{R}$ the fiberwise convex hull of $\mathcal{R}$.

17.3. Main lemma

17.3.1. (One-dimensional convex integration) *Let $\mathcal{R} \subset J^1(\mathbb{R}, \mathbb{R}^q)$ be an open differential relation and $F = (f, \varphi) : I \to \mathcal{R}$ be a short formal solution of $\mathcal{R}$. Then there exists a family of short formal solutions*

$$F_\tau = (f_\tau, \varphi_\tau) : I \to \mathcal{R},\ \tau \in [0, 1],$$

which joins $F_0 = F$ to a genuine solution F_1 such that

(a) *f_τ is (arbitrarily) C^0-close to f for all $\tau \in [0, 1]$;*

(b) *$F_\tau(0) = F(0)$ and $F_\tau(1) = F(1)$ for all $\tau \in [0, 1]$.*

◀ **Remark.** If the formal solution F is already genuine near ∂I then the homotopy F_τ can be chosen fixed near ∂I. Indeed, we can apply 17.3.1 to a smaller interval $[\delta, 1 - \delta] \subset I$. Strictly speaking, the constructed solution f_1 is only C^1-smooth at the points δ and $1 - \delta$. However, the relation $\mathcal{R}$ is open and hence f_1 can be approximated by a C^∞-smooth solution. ▶

17.3.2. (Corollary) *Let $\mathcal{R} \subset J^1(\mathbb{R}, \mathbb{R}^q)$ be an open* **ample** *differential relation. Then for any formal solution $F = (f, \varphi) : I \to \mathcal{R}$ which is genuine near ∂I there exists a homotopy of formal solutions, fixed near ∂I,*

$$F_\tau = (f_\tau, \varphi_\tau) : I \to \mathcal{R},\ \tau \in [0, 1],\ F_0 = F,$$

such that F_1 is a genuine solution of $\mathcal{R}$ over I and f_1 is (arbitrarily) C^0-close to f.

Indeed, the ampleness of the relation $\mathcal{R}$ implies that any formal solution of $\mathcal{R}$ is short, and hence we can apply 17.3.1. □

17.3.3. (Corollary) *Let $\mathcal{R} \subset J^1(\mathbb{R}, \mathbb{R}^q)$ be an open and fiberwise path-connected differential relation. Then the space of solutions $I \to \mathbb{R}^q$ of $\mathcal{R}$ is C^0-dense in the space of solutions of* $\operatorname{Conv}\mathcal{R}$. *If, in addition, $\mathcal{R}$ is ample and fiberwise non-empty then the space of solutions $I \to \mathbb{R}^q$ of $\mathcal{R}$ is C^0-dense in the space of all maps $I \to \mathbb{R}^q$.*

Indeed, the assumption that $\mathcal{R}$ is open and fiberwise path-connected guarantees the existence of a formal solution $F = (f, \varphi)$ of $\mathcal{R}$ for any solution f of $\operatorname{Conv}\mathcal{R}$, and hence we can apply 17.3.1. □

17.4. Proof of the main lemma

A. Flowers

An *abstract flower* S is a union of a finite number of copies $I_0, I_1, I_2, \dots$ of the interval $I = [0, 1]$ with their left ends identified to one point denoted 0_S. The interval $I_0 \subset S$ is called the *stem* of the flower, all the other intervals $I_1, \dots$ are called the *petals*. The ordering of petals is not essential for

Figure 17.2. An abstract flower.

our purpose. We will denote by ∂S the union of free ends of the petals I_i, $i = 1, \dots$ of the flower S.

A map $\psi : S \to \mathbb{R}^q$, and sometimes also its image $\Psi = \psi(S)$, will be called a *flower*. The parametrizing map $\psi : S \to \mathbb{R}^q$ is a union of paths $\psi_i : I \to \mathbb{R}^q$,

$\psi_i(0) = \psi(0_S)$. Let us point out that we do not assume the parametrizing map to be one-to-one. In particular, the map ψ may contract some of the petals or the stem into the point $\psi(0_S)$. Given a flower $\Psi = \psi(S)$ we set $a_i = \psi_i(1)$, $i = 1, \dots$ and $\partial\Psi = \psi(\partial S)$.

B. Reduction of Lemma 17.3.1 to its special case

Denote by D_ε the standard ε-ball in $\mathbb{R}^q$.

17.4.1. *It is sufficient to prove Lemma 17.3.1 for the case when*

- $\mathcal{R} = \mathbb{R} \times D^q_\varepsilon \times \Psi \subset J^1(\mathbb{R}, \mathbb{R}^q)$, *where* $\Psi = \psi(S) \subset \mathbb{R}^q$ *is a flower such that* $0 \in \operatorname{Int} \operatorname{Conv}(\partial\Psi)$*;*
- $F = (0, \varphi) : I \to \mathcal{R}$ *where* $\varphi \equiv \psi_0$*.*

◀ **Remarks**

1. Here and in the sequel we identify the section

$$\varphi : I \to I \times 0 \times \mathbb{R}^q \subset J^1(\mathbb{R}, \mathbb{R}^q)\,, \quad t \mapsto (t, 0, \bar{\varphi}(t))\,,$$

with the map $\bar{\varphi} : I \to \mathbb{R}^q$.

2. The relation $\mathcal{R} = \mathbb{R} \times D^q_\varepsilon \times \Psi$ is closed.

3. The formal solution $F = (0, \varphi)$ is automatically short. ▶

Proof. In the general case of Lemma 17.3.1 we can put $z = y - f(t)$ and consider, instead of $\mathcal{R} \sim \{\dot{y} \in \Omega(t, y)\}$, the "variation relation along $f(t)$":

$$\widetilde{\mathcal{R}} \sim \{\dot{z} \in \widetilde{\Omega}(t, z) = \Omega(t, z + f(t)) - \dot{f}(t)\}$$

and its short formal solution $(0, \varphi - \dot{f})$. For the further reduction we need the following

17.4.2. (Sublemma) *Let* $\mathcal{R} \subset J^1(\mathbb{R}, \mathbb{R}^q)$ *be an open differential relation and*

$$F = (0, \varphi) : I \to \mathcal{R}$$

be its short formal solution. Then there exists a number $\delta > 0$ *such that for any* $t_0 \in [0, 1 - \delta]$ *one can choose a flower*

$$\Psi = \Psi(t_0) \subset P_{t_0, 0} \simeq \mathbb{R}^q$$

with the properties

(a) $0 \in \operatorname{Int} \operatorname{Conv}(\partial\Psi)$*;*

(b) $\psi_0(t) = \varphi(t_0 + \delta t)$, $t \in I$*;*

(c) $[t_0, t_0 + \delta] \times D^q_\varepsilon \times \Psi \subset \mathcal{R}$ *for sufficiently small* $\varepsilon > 0$*.*

Proof. Let $t_0 \in I$. We can choose a finite set of points in $\operatorname{Conn}_{F(t_0)} \mathcal{R}$ such that 0 belongs to the interior of the convex hull of these points and then connect the base point $\varphi(t_0)$ with the chosen points by some paths $I \to \operatorname{Conn}_{F(t_0)} \mathcal{R}$. These paths are petals of our flower $\Psi = \Psi(t_0)$, while the path $\psi_0(t) = \varphi(t_0 + \delta t)$ is its stem. Then the flower Ψ has the properties (a) and (b). Using the openness of the relation $\mathcal{R}$ and the compactness of the interval I one can choose δ to satisfy (c). □

Using Sublemma 17.4.2 and an appropriate subdivision of the interval I we reduce Lemma 17.3.1 to the required special case. □

C. Uniform and weighted products of paths

Let us define a *(uniform) product*

$$p = p_1 \bullet \cdots \bullet p_k : I \to \mathbb{R}^q$$

of k paths $p_i : I \to \mathbb{R}^q$ in a natural way: p is a path of a particle which consecutively passes every $p_i(I)$ in $1/k$ seconds, i.e.:

$$p(t) = p_i(k[t - (i-1)/k])\,,\; t \in ((i-1)/n\,, i/n] \quad p(0) = p_1(0)\,.$$

If $p_i(1) = p_{i+1}(0)$ then p is continuous, otherwise p is only piecewise continuous.

We will need also a *weighted product* of paths. Let $\alpha_1 + \cdots + \alpha_k = 1\,, \alpha_i > 0\,.$ The product $p = p_1 \bullet \cdots \bullet p_k$ weighted by $(\alpha_1, \ldots, \alpha_k)$ is a trajectory of a particle which consecutively passes every $p_i(I)$ in α_i seconds. In particular, the uniform product of k paths is the product weighted by $(1/k, \ldots, 1/k)$. We allow α_i to be equal to *zero* if p_i is a *constant* path.

Given a path $p : I \to \mathbb{R}^q$, we will denote by p^N the uniform product

$$\underbrace{p \bullet \cdots \bullet p}_{n}$$

of n factors, and by $\int p(\sigma)\, d\sigma$ the path

$$t \mapsto \int_0^t p(\sigma)\, d\sigma\,.$$

The path $p : I \to \mathbb{R}^q$ is called *balanced* if $\int_0^1 p(\sigma)\, d\sigma = 0$.

In what follows we will need the two following evident properties of the *uniform* product:

17.4.3. ("Multiplicativity" of the integral) *Let $p_i : I \to \mathbb{R}^q$, $i = 1, \ldots, N$ be paths such that the paths $p_1, \ldots, p_{N-1}$ are balanced. Then*

$$\int (p_1 \bullet \cdots \bullet p_N)(\sigma)\, d\sigma = (1/N) \int p_1(\sigma)\, d\sigma \bullet \cdots \bullet \int p_N(\sigma)\, d\sigma\,.$$

17.4.4. (C^0-norm of the uniform product and convergence to zero) *Let $p_i : I \to \mathbb{R}^q$ be as in 17.4.3. Then*

$$\Big\| \int (p_1 \bullet \cdots \bullet p_N)(\sigma)\, d\sigma \Big\|_{C^0} =$$

$$= (1/N) \max \Big\{ \Big\| \int p_1(\sigma)\, d\sigma\, d\sigma) \Big\|_{C^0}, \ldots, \Big\| \int p_N(\sigma)\, d\sigma) \Big\|_{C^0} \Big\}\,.$$

In particular, if p is a balanced path then

$$\int p^N(\sigma)\, d\sigma \xrightarrow[N\to\infty]{C^0} 0\,.$$

D. Piecewise linear solution

Let us recall that we consider a special case of Lemma 17.3.1 described in 17.4.1.

By assumption we have $0 \in \operatorname{Int}\operatorname{Conv}(\partial\Psi)$, where $\partial\Psi = \{a_1, \ldots, a_k\}$, so we can write 0 as a convex combination $0 = \alpha_1 a_1 + \cdots + \alpha_k a_k$, $\alpha_i > 0$. Let $\delta : I \to \Psi$ be the product of *constant paths* $a_i : I \to a_i$, weighted by $(\alpha_1, \ldots, \alpha_k)$, so that δ is a piecewise constant discontinuous path. Then

$$\int_0^1 \delta(t)\, dt = 0.$$

In particular, δ is a *balanced path.*

Let $\varphi_1^\delta = \delta \bullet \delta \bullet \cdots \bullet \delta$ be the uniform product of N factors. We define a continuous piecewise linear path $f_1^\delta : I \to \mathbb{R}^q$ by the formula

$$f_1^\delta(t) = \int_0^t \varphi_1^\delta(\sigma)\, d\sigma$$

As follows from 17.4.4,

$$||f_1^\delta||_{C^0} = \frac{1}{N}\, ||g||_{C^0}\,,\ \text{where}\ g = \int \delta(\sigma)\, d\sigma\,.$$

Therefore, for $N > \frac{1}{\varepsilon}||g||_{C^0}$ the map f_1^δ is a piecewise linear solution of $\mathcal{R}$.

Now we want to realize the same idea in constructing a smooth solution. We will approximate the section φ_1^δ by a smooth section φ_1. In addition, we need to satisfy the boundary conditions for $F_1 = (f_1, \varphi_1)$.

E. Smooth solution

Let $\psi = \{\varphi, \psi_1, \dots, \psi_k\}$ be the parametrizing map for the flower Ψ. To ensure the smoothness of our further construction we will assume that $\psi_i(t) = \varphi(0)$ near $t = 0$ and $\psi_i(t) = a_i$ near $t = 1$.

Consider the product

$$\psi = \psi_1 \bullet a_1 \bullet \psi_1^{-1} \bullet \dots \bullet \psi_k \bullet a_k \bullet \psi_k^{-1}.$$

where the weights of constant paths a_i are equal to $(1-\rho)\alpha_i$ and the weights of all other paths are equal to $\rho/2k$.

For what follows we need to balance the loop ψ, i.e. we need the equality

$$\int_0^1 \psi(t)dt = 0.$$

This can be achieved by adjusting the weights of constant paths. Indeed, let

$$-d = \int_0^1 \psi(t)\,dt = \int_0^1 \psi(t)\,dt - \int_0^1 \delta(t)\,dt \in \mathbb{R}^q.$$

Then $||d|| < C\rho$, where

$$C = \max ||\psi(t)||,\ t \in I.$$

If ρ is sufficiently small then $d \in \mathrm{Int}\,\{(1-\rho)\,\Delta\}$ where $\Delta = \mathrm{Conv}\,\{a_1, \dots, a_k\}$. Here the multiplication by $(1-\rho)$ means the homothety centered at the origin. Note that

$$(1-\rho)\Delta = \mathrm{Conv}\,\{(1-\rho)a_1, \dots, (1-\rho)a_k\}.$$

Hence we can present $-d$ as the convex combination

$$-d = \widetilde{\alpha}_1(1-\rho)\,a_1 + \dots + \widetilde{\alpha}_k(1-\rho)\,a_k.$$

Therefore, if we assign the new weights $\widetilde{\alpha}_1(1-\rho),\ \dots, \widetilde{\alpha}_{q+1}(1-\rho)$ to the constant paths a_i, then the integral of $\psi(t)$ will be equal to 0.

Let $\varphi_1 = \underbrace{\psi \bullet \psi \bullet \dots \bullet \psi}_{N-1} \bullet \varphi$ be the uniform product of N factors, and f_1 be defined by the formula

$$f_1(t) = \int_0^t \varphi_1(\sigma)\,d\sigma.$$

Then 17.4.4 implies that

$$||f_1||_{C^0} = \frac{1}{N}\max\{||g||_{C^0}, ||h||_{C^0}\},$$

where

$$g = \int \psi(\sigma)\,d\sigma \ \text{ and } \ h = \int \varphi(\sigma)\,d\sigma.$$

If N is sufficiently large, then $F_1 = (f_1, \varphi_1)$ is a genuine solution of $\mathcal{R}$ and, moreover, F_1 satisfies the boundary conditions. The construction of the homotopy (f_τ, φ_τ) is straightforward: the linear homotopy $\tau f_1(t)$ consists of solutions of $\operatorname{Conv}\mathcal{R}$ and together with the canonical homotopies $\psi_i \circ \psi_i^{-1} \sim v_0 = \varphi(0)$ it gives us the required homotopy $F_\tau = (\tau f_1, \varphi_\tau)$ in $\mathcal{R}$. This finishes off the proof of the Main Lemma 17.3.1. □

17.5. Parametric version of the main lemma

17.5.1. (Parametric one-dimensional convex integration) *Let* $\mathcal{R} \subset I^l \times J^1(\mathbb{R}, \mathbb{R}^q)$ *be an open fibered differential relation* (see 6.2.E) *and*

$$F = F(p, t) = (f(p, t), \varphi(p, t)) : I^l \times I \to \mathcal{R}$$

be a fiberwise short formal solution of $\mathcal{R}$, *i.e. for each* $p \in I^l$ *the section*

$$F(p, t) : p \times I \to \mathcal{R}_p = \mathcal{R} \cap p \times J^1(\mathbb{R}, \mathbb{R}^q)$$

is a short formal solution of $\mathcal{R}_p$. *Suppose that* $f(p, t)$ **smoothly** *depends on* p *and consists of genuine solutions of* $\mathcal{R}_p$ *when* $p \in \mathcal{O}p\, I^l$. *Then there exists a homotopy of fiberwise short formal solutions*

$$F_\tau = F_\tau(p, t) = (f_\tau(p, t), \varphi_\tau(p, t)) : I^l \times I \to \mathcal{R},\ \tau \in [0, 1]\,,$$

which joins $F_0 = F$ *with a genuine solution* F_1 *of* $\mathcal{R}$ *such that for all* τ

(a) f_τ *is (arbitrarily)* C^0*-close to* f*;*

(b) $F_\tau(p, 0) = F(p, 0)$ *and* $F_\tau(p, 1) = F(p, 1)$ *for all* $p \in I^l$*;*

(c) F_τ *is constant for* $p \in \mathcal{O}p\,(\partial I^l)$*, and*

(d) *the first derivatives of* $f_1(p, t)$ *with respect to the parameter* p *are (arbitrarily)* C^0*-close to the respective derivatives of* $f(p, t)$*.*

◀ Remarks

1. If the formal solution F is already genuine near ∂I then the homotopy F_τ can be chosen fixed near ∂I . Indeed, we can apply 17.5.1 to a smaller set $I^l \times [\delta, 1 - \delta] \subset I^l \times I$.

2. We will use Lemma 17.5.1 in order to extend the convex integration of ordinary differential relations ($n = 1$) to the convex integration of partial differential relations ($n > 1$). The property (d) will be crucial for this goal. ▶

17.6. Proof of the parametric version of the main lemma

The proof will follow the same scheme as in the non-parametric version.

A. Fibered flowers

A *fibered* flower is a fibered (over I^l) map

$$\psi : I^l \times S \to I^l \times \mathbb{R}^q$$

(where S is an abstract flower), as well as the set

$$\Psi = \psi(I^l \times S) \subset I^l \times \mathbb{R}^q$$

parameterized by this map. Given a fibered flower Ψ we will denote by Ψ_p the flower

$$\psi(p \times S) \subset p \times \mathbb{R}^q \quad p \in I^l \, .$$

B. Reduction to a special case

17.6.1. *It is sufficient to prove 17.5.1 for the case when*

- *the fibered relation $\mathcal{R}$ consists of fibers*
$$\mathcal{R}_p = p \times \mathbb{R} \times D^q_\varepsilon \times \Psi_p \subset p \times J^1(\mathbb{R}, \mathbb{R}^q) \, , \quad p \in I^l \, ,$$
where $\Psi = \psi(I^l \times S) \subset I^l \times \mathbb{R}^q$ is a fibered flower such that $0 \in \operatorname{Int} \operatorname{Conv}(\partial \Psi_p)$ for each $p \in I^l$;
- *$F = (0, \varphi) : I^l \times I \to \mathcal{R}$ with $\varphi \equiv \psi_0$.*

◀ **Remark.** We identify here and further the fibered over I^l section

$$\varphi : I^l \times I \to I^l \times I \times 0 \times \mathbb{R}^q \subset I^l \times J^1(\mathbb{R}, \mathbb{R}^q) \, , \quad (p,t) \mapsto (p, t, 0, \bar{\varphi}(p,t)) \, ,$$

with the fibered map

$$I^l \times I \to I^l \times \mathbb{R}^q \, , \quad (p,t) \to (p, \bar{\varphi}(p,t)) \, . \; ▶$$

Proof. As in the non-parametric case we may assume that $f \equiv 0$. To make the further reduction we need the following

17.6.2. (Sublemma) *Let $\mathcal{R} \subset I^l \times J^1(\mathbb{R}, \mathbb{R}^q)$ be an open fibered differential relation and*

$$F = (0, \varphi) : I^l \times I \to \mathcal{R}$$

be a fiberwise short formal solution of $\mathcal{R}$. There exists a number $\delta > 0$ such that for any $t_0 \in [0, 1 - \delta]$ one can choose a fibered flower

$$\Psi = \Psi(t_0) \subset I^l \times P_{t_0,0} \simeq I^l \times \mathbb{R}^q$$

such that for each $p \in I^l$

(a) $0 \in \operatorname{Int}\operatorname{Conv}(\partial\Psi_p)$;

(b) $\psi_0(p,t) = \varphi(p, t_0 + \delta t)$, $t \in I$;

(c) $p \times [t_0, t_0 + \delta] \times D^q_\varepsilon \times \Psi_p \subset \mathcal{R}_p$ *for sufficiently small* $\varepsilon > 0$.

Proof. Let $t_0 \in I$. First take a fibered flower which consists of just its stem parametrized by the map $\psi_0 : (p,t) \mapsto \varphi(p, t_0 + \delta t)$ (the number δ will be chosen later).

For every fixed $p_0 \in I^l$ we can choose a flower Ψ_{p_0} as in Sublemma 17.4.2, and using the openness of $\mathcal{R}$ extend Ψ_{p_0} over a neighborhood U of $p_0 \in I^l$ such that for all $p \in U$

- $\psi_i(p,t)$ are paths in $\operatorname{Conn}_{F(p,t_0)} \mathcal{R}$, and
- $0 \in \operatorname{Int}\operatorname{Conv}(\partial\Psi_p)$.

Hence, we can chose a finite covering of I^l by open sets U_j, $j = 1, \dots, L$, such that over every U_j we have, as above, a flower fibered over U_j, which we denote by Ψ^{U_j}. Suppose that its petals are parametrized by the maps

$$\psi_i^{U_j} : U_j \times I \to \mathcal{R},\ i = 1, \dots, N_j\,,\ \ j = 1, \dots, L\,.$$

Let $U'_j \subset U_j\,, j = 1, \dots, L,$ be slightly smaller open sets such that

$$\overline{U}'_j \subset U_j \ \text{ and } \ \bigcup_1^L U'_j \supset I^l\,.$$

For every $j = 1, \dots, L$ choose a cut-off function $\beta^j : I^l \to [0,1]$ which is equal to 1 on U'_j and equal to 0 on $I^l \setminus U_j$, and for $i = 1, \dots N_j$ set

$$\psi_i^j(p,t) = \psi_i^{U_j}(p, \beta^j(p)\,t) \ \text{ for } \ p \in U_j \ \text{ and}$$

$$\psi_i^j(p,t) = \varphi(p, t_0) \ \text{ for } \ p \in I^l \setminus U_j\,,$$

The fibered (over I^l) flower with the stem $\psi_0(p,t) = \varphi(p, t_0 + t\delta, p)$ and petals parameterized by all the maps ψ_i^j for all $i = 1, \dots, N_j$, $j = 1, \dots, L$, satisfies the properties (a) and (b). Therefore using the the openness of the relation $\mathcal{R}$ and the compactness of the interval I, one can choose δ to satisfy (c). □

Using Sublemma 17.6.2 and an appropriate subdivision of the interval I one can reduce the Parametric Main Lemma 17.5.1 to the required special case. □

C. Convex decomposition of a section

Let Ψ be a flower fibered over I^l. Let us set $a_i(p) = \psi_i(1)$, $i = 1, \dots, N$, $\Delta_p = \operatorname{Conv}\partial\Psi_p$. Let $d : p \mapsto \operatorname{Int}\Delta_p$ be a section over I^l. Then there exist functions $\alpha_i : I^l \to [0,1]$, $i = 1, \dots, N$, such that

$$\alpha_1(p) + \cdots + \alpha_N(p) = 1 \text{ and } \alpha_1(p)a_1(p) + \cdots + \alpha_N(p)a_N(p) = d(p)\,.$$

Indeed, we can construct such a set of functions locally over a neighborhood of each point $p \in I^l$ and then globalize the construction using a partition of unity.

D. Construction of the homotopy F_τ

We can apply the proof of the Main Lemma parametrically, working with convex decompositions of the sections $\mathbf{0}$ and $d(p)$ instead of convex decompositions of the vectors 0 and d, and with weights $\alpha_i(p)$, $\widetilde{\alpha}_i(p)$ which depend on the parameter p. This way we construct a family of (balanced) paths $\psi(p,t)$ and then a family of functions $\varphi_\tau(p,t)$ such that the respective family of sections F_τ satisfies properties (a) and (b). In order to satisfy property (c) it is sufficient to set $F_\tau := F_{\beta(p)\tau}$, where the function $\beta : I^l \to [0,1]$ is equal to 0 near ∂I^l and equal to 1 on a slightly smaller cube $I_1^l \subset I^l$, so that $F_0(p,t) = F(p,t)$ is a genuine solution of $\mathcal{R}_p$ for all $p \in I^l \setminus I_1^l$.

Let us now turn to property (d), which is specific for the parametric case and which is crucial for further generalizations. For $F = (0, \varphi)$ this property means that the derivatives $\partial_p f_1(t,p)$ are arbitrarily close to 0. Take the uniform product

$$\varphi_1(p,*) = \psi(p) \bullet \cdots \bullet \psi(p) \bullet \varphi(p)$$

of N factors and set

$$f_1(p,t) = \int_0^t \varphi_1(p,\sigma)\,d\sigma\,.$$

Then

$$\partial_p f_1(p,t) = \partial_p \int_0^t \varphi_1(p,\sigma)\,d\sigma = \int_0^t \partial_p \varphi_1(p,\sigma)\,d\sigma$$

where

$$\partial_p \varphi_1(p,*) = \partial_p \psi(p) \bullet \cdots \bullet \partial_p \psi(p) \bullet \partial_p \varphi(p)\,.$$

The path $\partial_p \psi(p)$ is balanced because

$$\int_0^1 \partial_p \psi(p,\sigma) = \partial_p \int_0^1 \psi(p,\sigma)d\sigma = \partial_p 0 \equiv 0\,.$$

Hence, according to 17.4.4 we have

$$||\partial_p f_1||_{C^0} = \frac{1}{N} \max\{||\partial_p g||_{C^0}, ||\partial_p h||_{C^0}\}\,,$$

where

$$g(p,t) = \int_0^t \psi(p,\sigma)\,d\sigma\,, \text{ and } h(p,t) = \int_0^t \varphi(p,\sigma)\,d\sigma$$

Therefore, $\partial_p f_1 \to 0$ when $N \to \infty$.

E. Remark.

The same proof is also valid in the case when F consists of genuine solutions over a neighborhood of a closed subset $A \subset \partial D^l$ (instead of the whole ∂D^l). In this case the homotopy F_τ is constant for $p \in A$ (instead of $p \in \partial D^l$).

Homotopy Principle for Ample Differential Relations

18.1. Ampleness in coordinate directions

A *coordinate principal subspace* in a fiber $M_{q\times n} = (\mathbb{R}^q)^n$ of the fibration

$$J^1(\mathbb{R}^n, \mathbb{R}^q) = J^0(\mathbb{R}^n, \mathbb{R}^q) \times M_{q\times n} \to J^0(\mathbb{R}^n, \mathbb{R}^q) = \mathbb{R}^n \times \mathbb{R}^q$$

is any q-dimensional affine subspace parallel to one of the factors $\mathbb{R}^q$ in the product $(\mathbb{R}^q)^n$ or, what is the same, the set of all $q \times n$ matrices with fixed $(n-1)$ columns. Thus for every point $z \in J^1(\mathbb{R}^n, \mathbb{R}^q)$ there are n coordinate principal subspaces $P^1(z), \dots, P^n(z)$ which go through z. A particular coordinate principal subspace $P^i(z)$ over a point $(x, f(x)) \in J^0(\mathbb{R}^n, \mathbb{R}^q)$ can be interpreted as *the space of all possible vector-derivatives $\partial_{x_i} f$ under the condition that all the other vector-derivatives $\partial_{x_j} f$, $j \neq i$, are fixed.*

Let us recall that a set $\Omega \subset P$ where P is an affine space is called *ample* if the convex hull of each path-connected component of Ω is P or if Ω is empty.

A differential relation $\mathcal{R} \subset J^1(\mathbb{R}^n, \mathbb{R}^q)$ is called *ample in the coordinate directions* if $\mathcal{R}$ intersects all coordinate principal subspaces along ample sets.

◀ **Examples**

1. If $n < q$ then the immersion relation $\mathcal{R}_{\mathrm{imm}} \subset J^1(\mathbb{R}^n, \mathbb{R}^q)$, which consists of all matrices of rank n, is ample in the coordinate directions. Indeed, for any $z = (x, y, a) \in \mathcal{R}_{\text{imm}}$ and any coordinate principal subspace $P = P^i(z)$ we have $P \cap \mathcal{R} = P \setminus L$ where L is an $(n-1)$-dimensional linear subspace in $P \simeq \mathbb{R}^q$ spanned by all the columns of the matrix a except the i-th column. The codimension of L in P is less than 1 and hence $\text{Conv}\,(P \cap \mathcal{R}) = P$.

2. If $n \geq q$ then the submersion relation $\mathcal{R}_{\text{sub}} \subset J^1(\mathbb{R}^n, \mathbb{R}^q)$, which consists of all matrices of rank q, is not ample in the coordinate directions. Indeed, for any $z = (x, y, a) \in \mathcal{R}_{\text{sub}}$ and any coordinate principal subspace $P = P^i(z)$ we have $P \cap \mathcal{R} = P \setminus L$ where L is a $(q-1)$-dimensional linear subspace in $P \simeq \mathbb{R}^q$ spanned by all the columns of the matrix a except the i-th column. Therefore $P \cap \mathcal{R}$ consists of two open half-spaces, and thus is not ample. ▶

◀ **Exercise.** Prove that the differential relation $\mathcal{R}_{k-\text{mers}} \subset J^1(\mathbb{R}^n, \mathbb{R}^q)$ which consists of all matrices of the rank $\geq k$ is ample if $k < q$. ▶

A singularity

$$\Sigma \subset J^1(\mathbb{R}^n, \mathbb{R}^q) = J^0(\mathbb{R}^n, \mathbb{R}^q) \times M_{q \times n}$$

is called *thin in the coordinate directions* if it intersects all the coordinate principal subspaces along stratified subsets of codimension ≥ 2. In this case the complement $\mathcal{R} = J^1(\mathbb{R}^n, \mathbb{R}^q) \setminus \Sigma$ is a differential relation ample in the coordinate directions.

18.2. Iterated convex integration

18.2.1. (Convex integration over a cube) *Let $\mathcal{R} \subset J^1(\mathbb{R}^n, \mathbb{R}^q)$ be an open differential relation ample in the coordinate directions and*

$$F = (f, \varphi) : I^n \to \mathcal{R} \subset J^0(\mathbb{R}^n, \mathbb{R}^q) \times M_{q \times n}$$

a formal solution of $\mathcal{R}$ which is a genuine solution near ∂I^n. Then there exists a homotopy of formal solutions

$$F_\tau = (f_\tau, \varphi_\tau) : I^n \to \mathcal{R},\ \tau \in [0, 1]\,,$$

which joins $F_0 = F$ with a genuine solution F_1 of $\mathcal{R}$ such that for all τ

- *f_τ is (arbitrarily) C^0-close to f;*
- *F_τ coincides with F near ∂I^n.*

Proof. Let $(\varphi^1, \dots, \varphi^n)$ be the *columns* of the matrix φ. We will integrate the formal solution $F = (f, \varphi)$ coordinate-wise, using Lemma 17.5.1.

At the first step we consider the cube I^n as a family of intervals $I \times p$, $p \in I^{n-1}$, parallel to the x_1-axis. Let us form a relation $\mathcal{R}^1 \subset I^{n-1} \times J^1(\mathbb{R}, \mathbb{R}^q)$, fibered over I^{n-1}, which is defined over a small neighborhood of the graph of the section f in $I^n \times \mathbb{R}^q$ in the following way. For $t = x_1$, $p = (x_2, \dots, x_n)$ we define the set

$$\Omega_p(f(t,p)) = \mathcal{R}^1_p \cap f(t,p) \times \mathbb{R}^q \subset p \times J^1(\mathbb{R}, \mathbb{R}^q)$$

as the path-connected component of $\mathcal{R} \cap P^1(F(t,p))$ which contains the point $F(t,p)$. Here we use the canonical identification $P^1(F(t,p)) \simeq \mathbb{R}^q$. In order to expand $\mathcal{R}_p$ to a small neighborhood of the graph of f one can slightly decrease the (open!) sets $\Omega_p(f(t,p))$ in such a way that the new sets are still ample and for a sufficiently small ε the product $D^q_\varepsilon(f(x)) \times \Omega_p(f(x)) \subset J^1(\mathbb{R}, \mathbb{R}^q)$ is contained in $\mathcal{R}$. Here $D^q_\varepsilon(f(x))$ denotes the ε-ball in $\mathbb{R}^q$ centered at $f(x)$.

Now we can apply Lemma 17.5.1 to the fibered relation $\mathcal{R}^1$ and its fibered formal solution $(f(t,p), \varphi^1)$, which is automatically short because the relation $\mathcal{R}$ is ample. As a result we will have a genuine solution $(f^1(t,p), \partial_t f^1(t,p))$ of the relation $\mathcal{R}^1$ and hence the new formal solution

$$F^1 = (f^1;\, \partial_{x_1} f^1,\, \varphi^2, \dots, \varphi^n)$$

of the relation $\mathcal{R}$. This formal solution is homotopic to F in $\mathcal{R}$, coincides with f near ∂I^n, while the section f^1 is C^0-close to f. But what is most important, the section F^1 *is holonomic with respect to the coordinate* x_1.

At the second step we consider the cube I^n as a family of intervals parallel to the axis x_2, form a relation $\mathcal{R}^2$, fibered over I^{n-1}, and construct a new formal solution

$$F^2 = (f^2;\, \partial_{x_1} f^1,\, \partial_{x_2} f^2,\, \varphi^3, \dots, \varphi^n)$$

of $\mathcal{R}$. This formal solution is holonomic with respect to the coordinate x_2, i.e. $\varphi^2 = \partial_{x_2} f^2$. According to property 17.5.1 (d) the section $\partial_{x_1} f^2$ is (arbitrarily) C^0-close to the section $\partial_{x_1} f^1$ and hence we can deform the formal solution F^2 by a linear homotopy in $\mathcal{R}$ into a formal solution

$$\widetilde{F}^2 = (f^2;\, \partial_{x_1} f^2,\, \partial_{x_2} f^2,\, \varphi^3, \dots, \varphi^n)$$

which is holonomic with respect to both coordinates, x_1 and x_2.

Thus using the parametric version of one-dimensional convex integration we can realize the following chain of homotopies (each arrow denotes a homotopy; $f_0 = f$):

$$\begin{gathered}(f_0;\, \varphi^1, \varphi^2, \dots, \varphi^n) \to (f^1;\, \partial_{x_1} f^1, \varphi^2, \dots, \varphi^n) \to \dots \\ \to (f^i;\, \partial_{x_1} f^i, \dots, \partial_{x_i} f^i, \varphi^{i+1}, \varphi^{i+2}, \dots, \varphi^n) \to \\ \to (f^{i+1};\, \partial_{x_1} f^{i+1}, \dots, \partial_{x_{i+1}} f^{i+1}, \varphi^{i+2}, \dots, \varphi^n) \to \\ \dots \to (f^n;\, \partial_{x_1} f^n, \dots, \partial_{x^n} f^n)\,.\end{gathered}$$

Property 17.5.1 (d) is crucial here: it allows us to realize each homotopy in the chain as a homotopy in $\mathcal{R}$. □

◀ **Remark** The same proof is valid in the parametric case (for families of sections over the cube I^n) and also in the case when F is a genuine solution near a neighborhood of a closed subset $A \subset \partial I^n$ (instead of the whole ∂I^n). In this case the homotopy F_τ can be chosen constant near A (instead of ∂I^n). ▶

18.2.2. (Corollary: h-principle for differential relations over a cube) *Let $\mathcal{R} \subset J^1(\mathbb{R}^n, \mathbb{R}^q)$ be an open differential relation over the cube I^n ample in the coordinate directions. Then all forms of the relative h-principle hold for $\mathcal{R}$ over the pair $(I^n, \partial I^n)$ and also over the pair (I^n, A), where $A \subset \partial I^n$ is any closed subset.*

18.3. Principal subspaces and ample differential relations in $X^{(1)}$

Let $p : X \to V$ be a fibration. Let us recall that the fiber

$$E_x = (p_0^1)^{-1}(x), \quad x \in X ,$$

of the projection

$$p_0^1 : X^{(1)} \to X^{(0)} = X$$

can be identified with $\mathrm{Hom}(T_v V, \mathrm{Vert}_x)$, where $v = p(x)$ and Vert_x is the tangent space to the fiber of the fibration $p : X \to V$ at the point $x \in X$. Given a hyperplane $\tau \subset T_v V$ and a linear map $l : \tau \to \mathrm{Vert}_x$, let us denote by P_τ^l an affine subspace of E_x defined as

$$P_\tau^l = \{L \in \mathrm{Hom}(T_v V, \mathrm{Vert}_x) \mid \ L|_\tau = l\} .$$

Affine subspaces of E_x of this type are called *principal.* Note that the direction of the principal subspace P_τ^l is determined by the hyperplane $\tau \in T_v V$, and thus the principal *directions* at a given fiber E_x are parameterized by the projective space $P(T_v^* V) \cong \mathbb{R}P^{n-1}$, where $n = \dim V$.

Alternatively the 1-jet space $X^{(1)}$ can be considered as the space of all non-vertical n-planes ξ in TX. In this interpretation principal subspaces are non-vertical n-planes which contain a fixed non-vertical $(n-1)$-plane in $T_x X$.

If $X = \mathbb{R}^n \times \mathbb{R}^q \to \mathbb{R}^n = V$ is a trivial fibration, so that we have $X^{(1)} = J^1(\mathbb{R}^n, \mathbb{R}^q)$, then our previously defined *coordinate principal subspaces* are the principal subspaces directed by hyperplanes $\{x_i = \mathrm{const}\}$ in $\mathbb{R}^n$.

Any principal subspace in $X^{(1)}$ has a natural *affine* structure, but no natural *linear* structure, even in the case of a trivial fibration $X = V \times W \to V$.

A differential relation $\mathcal{R} \subset J^1(\mathbb{R}^n, \mathbb{R}^q)$ is called *ample* if $\mathcal{R}$ intersects all principal subspaces along ample sets.

◀ **Remark.** The ampleness in *coordinate* directions looks like a less restrictive property than ampleness. However, *for* Diff V*-invariant relations the two notions of ampleness coincide.* In fact we do not know any geometrically interesting examples when the less restrictive notion of ampleness is satisfied but the other one is not. ▶

◀ **Examples.** The immersion relation $\mathcal{R}_{\mathrm{imm}} \subset X^{(1)}$ is ample if $n < q$. The submersion relation $\mathcal{R}_{\mathrm{subm}} \subset X^{(1)}$ is not ample. The k-mersion relation $R_{k-\mathrm{mers}} \subset X^{(1)}$ is ample if $k < q$. ▶

A singularity $\Sigma \subset X^{(1)}$ is called *thin* if for any $a \in \Sigma$ and any principal subspace P through the point a the intersection $P \cap \Sigma$ is a manifold or, more generally, a stratified subset of codimension ≥ 2 in P. If Σ is thin than the complementary differential relation $\mathcal{R} = X^{(1)} \setminus \Sigma$ is ample.

18.4. Convex integration of ample differential relations

18.4.1. (Homotopy principle for ample differential relations) *Let $\mathcal{R} \subset X^{(1)}$ be an open ample differential relation. Then all forms of the h-principle hold for $\mathcal{R}$.*

Proof of Theorem 18.4.1. The induction over skeleta of a triangulation of the base V reduces the h-principle 18.4.1 to the relative h-principle 18.2.2. □

18.4.2. (Corollary: removal of a thin singularity) *Let $\Sigma \subset X^{(1)}$ be a thin singularity. Then all forms of the h-principle hold for Σ-non-singular sections of $X^{(1)}$.*

In particular, the k-mersion relation $\mathcal{R}_{k-\mathrm{mers}}$ is ample if $k < q$ (the respective singularity $\Sigma = X^{(1)} \setminus \mathcal{R}_{k-\mathrm{mers}}$ is thin) and hence 18.4.1 implies the h-principle for k-mersions $V \to W$, $k < \dim W$. The case $k = n$ gives us the h-principle for immersions $V \to W$, $\dim V < \dim W$. Note that the h-principle for *submersions* (of open manifolds) does not follow from 18.4.1 because the relation $\mathcal{R}_{\mathrm{subm}}$ is not ample.

◀ **Remarks**

1. We will discuss further applications of the convex integration method in Chapters 19 and 20 below. In all these examples we will verify the ampleness of the respective differential relations. The ampleness will then imply *all forms* of the h-principle (relative, parametric, C^0-dense).

2. Suppose that the manifold V is covered by coordinate charts U_i, $i = 1, \dots, N$. We can relax the ampleness condition for the differential relations

$\mathcal{R} \subset X^{(1)}$ in Theorem 18.4.1 by requiring instead that over each neighborhood U_i, $i = 1, \dots, N$, the relation $\mathcal{R}$ is ample only in respective coordinate directions. However, as we already pointed out above, we do not know any interesting examples where this formulation of the h-principle 18.4.1 is really stronger. ►

Chapter 19

Directed Immersions and Embeddings

19.1. Criterion of ampleness for directed immersions

Let us recall the definition of *directed immersions* from 4.5.

Let $\mathrm{Gr}_n W$ be the Grassmanian bundle of tangent n-planes to a q-dimensional manifold W, $q > n$. Let V be an n-dimensional manifold. Given a monomorphism $F : TV \to TW$, we denote by $\mathrm{G}F$ the corresponding map $V \to \mathrm{Gr}_n W$. Let $A \subset \mathrm{Gr}_n W$ be an arbitrary subset. An immersion $f : V \to W$ is called *A-directed* if $\mathrm{G}df$ sends V into A. If V is an *oriented* manifold then we can also consider A-directed immersions where A is an arbitrary subset in the Grassmanian $\widetilde{\mathrm{Gr}}_n W$ of *oriented* tangent n-planes to a q-dimensional manifold W.

Given a subset $A \subset \mathrm{Gr}_n W$ we will denote by $\mathcal{R}_A$ the differential relation in $\mathcal{R}_{\mathrm{imm}} \subset J^1(V, W)$ which corresponds to A-directed immersions $V \to W$, by A_w the fiber $A \cap \mathrm{Gr}_n(T_w W)$, $w \in W$, and by $\mathrm{Gr}_{n-1} A_w$ the set

$$\bigcup_{L \in A_w} \mathrm{Gr}_{n-1}(L) \subset \mathrm{Gr}_{n-1} T_w W\,.$$

19.1.1. (Ampleness criterion) *The relation $\mathcal{R}_A$ is ample if and only if for every $w \in W$ and every $S \in \mathrm{Gr}_{n-1} A_w$ the set*

$$\Omega_S = \{v \in T_w W \mid \mathrm{Span}\{S, v\} \in A_w\} \subset T_w W$$

is ample.

Proof. Let us check that the above condition implies the ampleness of $\mathcal{R}_A$. Note that any principal subspace is a coordinate principal subspace for a certain local coordinate system, and by choosing a local coordinate system we can work in $J^1(\mathbb{R}^n, \mathbb{R}^q)$. For $s = (x, y, a) \in \mathcal{R}_A$ let $P = P^i(a)$ be a coordinate principal subspace over (x, y). The subspace P can be canonically identified with $T_y\mathbb{R}^q$. The intersection $P \cap \mathcal{R}_A$ consists of all vectors $v \in T_y\mathbb{R}^q$ such that $\operatorname{Span}\{S, v\} \in A$, where $S \subset T_y\mathbb{R}^q$ is the $(n-1)$-dimensional linear subspace spanned by all the columns of the matrix a except the i-th column. Thus this intersection is equal to Ω_S, and hence ample. The opposite implication follows from the $\operatorname{Diff} V$-invariance of the relation $\mathcal{R}_A$. □

Condition 19.1.1 can be reformulated in the following way:

19.1.2. *The relation $\mathcal{R}_A$ is ample if and only if for every $w \in W$ and every $S \in \widehat{A}_w$ the set*

$$\Omega'_S = \{v \in S^\perp \,|\, \operatorname{Span}\{S, v\} \in A\} \subset S^\perp ,$$

where $S^\perp$ is the orthogonal complement to $S \subset T_wW$, is ample.

For an *oriented* manifold V we can consider

$$\widetilde{\operatorname{Gr}}_{n-1}A_\omega = \bigcup_{L \in A_w} \widetilde{\operatorname{Gr}}_{n-1}(L) \subset \widetilde{\operatorname{Gr}}_{n-1}T_wW$$

and the oriented version of the above criterions.

◄ **Exercise.** Suppose that V is oriented and $W = \mathbb{R}^{n+1}$. Prove that the oriented version of Condition 19.1.2 means that for every $a \in A \subset \widetilde{\operatorname{Gr}}_nW = \mathbb{R}^{n+1} \times S^n$ and every great circle $S^1 \subset y \times S^n$ through a the intersection $S^1 \cap A$ contains an arc of length $> \pi$. ►

19.2. Immersions into almost symplectic and almost complex manifolds

A. Directed immersions into almost symplectic manifolds

Let us recall that an almost symplectic structure on a manifold W of dimension $q = 2k$ is a non-degenerate but not necessarily closed 2-form ω. One can define *symplectic, Lagrangian, isotropic* and *coisotropic* immersions $V \to (W, \omega)$ as A-directed immersions where $A \subset \operatorname{Gr}_nW$ is the respective (symplectic, Lagrangian, etc.) Grassmannian of n-planes tangent to W. We denote the corresponding differential relations in $J^1(V, W)$ by $\mathcal{R}_{\mathrm{symp}}$, $\mathcal{R}_{\mathrm{Lag}}$, $\mathcal{R}_{\mathrm{isot}}$ and $\mathcal{R}_{\mathrm{coisot}}$.

The relation $\mathcal{R}_{\mathrm{symp}}$ is open, while $\mathcal{R}_{\mathrm{Lag}}$, $\mathcal{R}_{\mathrm{isot}}$ and $\mathcal{R}_{\mathrm{coisot}}$ are closed. We can take open neighborhoods of these relations considering for any ε, $0 < \varepsilon <$

$\pi/2$, *ε-Lagrangian, ε-isotropic* and *ε-coisotropic* immersions $V \to (W, \omega)$ as A^ε-directed immersions, where A^ε is the ε-neighborhood of the respective set A in $\mathrm{Gr}_n W$. We assume here that W is endowed with a Riemannian metric. The respective differential relations in $J^1(V, W)$ will be denoted by $\mathcal{R}^\varepsilon_{\mathrm{Lag}}$, $\mathcal{R}^\varepsilon_{\mathrm{isot}}$ and $\mathcal{R}^\varepsilon_{\mathrm{coisot}}$.

◀ **Exercise.** Prove that the relations $\mathcal{R}^\varepsilon_{\mathrm{Lag}}$ and $\mathcal{R}^\varepsilon_{\mathrm{isot}}$ are ample and hence all forms of the h-principle hold for ε-Lagrangian and ε-isotropic immersions $V \to (W, \omega)$. ▶

◀ **Remark.** The proof of the h-principle for isotropic and in particular Lagrangian immersions into *symplectic* manifolds which we gave in Part III of the book fails when we try to generalize it for immersions into *almost symplectic* manifolds. On the other hand, the ampleness which is needed for the application of convex integration does not depend *a priori* on the closeness of the form ω and hence convex integration equally works for a symplectic or almost symplectic target manifold W. ▶

◀ **Exercise.** Show that the relations $\mathcal{R}_{\mathrm{symp}}$, $\mathcal{R}^\varepsilon_{\mathrm{coisot}}$ and $\mathcal{R}^\varepsilon_{\mathrm{isosymp}}$ are not ample. ▶

◀ **Problems.** Is there any form of the h-principle for

(a) Lagrangian and isotropic immersions into an almost symplectic manifold?

(b) isosymplectic immersions between almost symplectic manifolds?

(c) coisotropic and isometric coisotropic immersions into almost symplectic manifolds?

Comments to the problems. We do not know the answer to most of these questions. However, it seems to us that the *positive answer* to (a) should not be difficult to prove. The answer to (b) is obviously negative in the most general set-up, even locally in a neighborhood of a point. On the other hand, when the source manifold is 2-dimensional, then Theorem 16.4.3 remains true even when the target manifold is only almost symplectic. The problem in (b) is to find the conditions under which some kind of the h-principle may hold. Theorem 16.5.1 implies the h-principle for *isometric* coisotropic immersions into symplectic manifolds. We do not know whether it remains true when the structure on the target manifold is not integrable.

B. Directed immersions into almost complex manifolds

Let us recall that a subspace $S \subset \mathbb{C}^n$ is called

- *complex*, if $iS = S$;

- *real* or *totally real*, if $S \cap iS = 0$; in other words a subspace is totally real if it contains no complex subspaces of positive dimension;
- *co-real*, if $S + iS = L$.

Let (W, J) be an almost complex manifold of (real) dimension $q = 2k$. One can define *complex* and *real* immersions $V \to (W, J)$ as A-directed immersions where $A \subset \mathrm{Gr}_n W$ are the respective Grassmannians A_{comp}, A_{real} and A_{coreal} of complex, real and co-real n-planes in TW. Denote the corresponding differential relations in $J^1(V, W)$ by $\mathcal{R}_{\mathrm{comp}}$, $\mathcal{R}_{\mathrm{real}}$, and $\mathcal{R}_{\mathrm{coreal}}$.

For any ε, $\pi/2 > \varepsilon > 0$, we also define ε-*complex* immersions $V \to (W, \omega)$ as $A^{\varepsilon}_{\mathrm{comp}}$-directed immersions where $A^{\varepsilon}_{\mathrm{comp}}$ is the ε-neighborhood of A_{comp} in $\mathrm{Gr}_n W$. The corresponding differential relation in $J^1(V, W)$ is denoted by $\mathcal{R}^{\varepsilon}_{\mathrm{comp}}$.

◀ **Exercise.** Prove that the differential relation $\mathcal{R}^{\varepsilon}_{\mathrm{comp}}$ is not ample. ▶

19.2.1. (Gromov [Gr86]) *The relations $\mathcal{R}_{\mathrm{real}}$ and $\mathcal{R}_{\mathrm{coreal}}$ are ample and hence all forms of the h-principle hold for real and co-real immersions $V \to (W, J)$.*

Proof. Let $n \le k$. For a particular $(n-1)$-dimensional subspace $S \subset L \in A_{\mathrm{real}}$ over a point $w \in W$ we have

$$\Omega_S = T_w W \setminus (S + iS)\,,$$

where $\dim (S + iS) = 2n - 2 \le 2k - 2$, and hence $\mathcal{R}_{\mathrm{real}}$ is the complement of a thin singularity.

For $n > k$ the set Ω_S is the complement of S if $S + iS = T_w W$ or is a complement of $S + iS$ if $S + iS \neq T_w W$. In both cases the codimension of the singularity is ≥ 2 and hence $\mathcal{R}_{\mathrm{coreal}}$ is the complement of a thin singularity as well. □

19.3. Directed embeddings

19.3.1. (Directed embeddings) *Suppose that $A \subset \mathrm{Gr}_n W$ is an open subset and the corresponding (open) differential relation $\mathcal{R}_A \subset J^1(V, W)$ is ample. Then every embedding $f_0 : V \to W$ whose tangential lift*

$$G_0 = \mathrm{G}df_0 : V \to \mathrm{Gr}_n W$$

is homotopic over V to a map $G_1 : V \to A$ can be isotoped to an A-directed embedding $f_1 : V \to W$. Moreover, such an isotopy $f_t : V \to W$ can be chosen arbitrarily C^0-close to the constant isotopy.

Here homotopy *over* V means that the underlying homotopy $g_t : V \to W$ for G_t is constant (i.e. G_t is a *tangential* homotopy, as in 4.4). In fact, the theorem is also true, with an obvious modification of the proof, in the case when G_t is a homotopy *over embeddings* $g_t : V \to W$. In this case f_t can be chosen arbitrarily C^0-close to g_t. We restrict ourselves to the case $g_t = f_0$ only to clarify the main idea of the proof.

Theorem 19.3.1 also holds (with the same proof) in the relative and parametric versions. Here is, for example, the parametric version of Theorem 19.3.1.

19.3.2. (Families of directed embeddings) *Suppose that* $f^p : V \to W$, $p \in D^l$, *is a family of embeddings which are A-directed for* $p \in \partial D^l$, *and* $G_t^p : V \to \mathrm{Gr}_n W$, $t \in [0,1]$, *is a homotopy of tangent lifts, constant over* ∂D^l, *such that* $G_0^p = \mathrm{G}df^p$ *and* G_1^p *sends* V *to* A *for each* $p \in D^l$. *Then there exists a family of isotopies* $f_t^p : V \to W$, *constant over* ∂D^l, *such that* f_1^p *is an A-directed embedding for all* $p \in D^l$.

Proof of Theorem 19.3.1. Assuming that the manifold W is endowed with a Riemannian metric we can cover the homotopy $\bar{G}_t$ by a homotopy of fiberwise isomorphisms $\Phi_t : T(W) \to T(W)$, $\mathrm{bs}\,\Phi_t = \mathrm{Id}_W$. Then the existence of the required isotopy f_t follows from

19.3.3. *Let* $A \subset \mathrm{Gr}_n W$ *be an open subset such that the corresponding differential relation* $\mathcal{R}_A \subset J^1(V,W)$ *is ample. Let* $\Phi^t : TW \to TW$, $t \in [0,1]$, *be a homotopy of fiberwise isomorphisms such that* $\mathrm{bs}\,\Phi_t = \mathrm{Id}_W$ *for all* t. *Then for every A-directed embedding* $f_0 : V \to W$ *there exists an isotopy* $f_t : V \to W$, $t \in [0,1]$, *such that* f_1 *is an* A^1*-directed embedding, where* $A^1 = \Phi_*^1 A$. *Moreover, such an isotopy* f_t *can be chosen arbitrarily* C^0*-close to the constant isotopy.*

Proof of Theorem 19.3.3. We begin with the following lemma which is an immediate corollary of the Ampleness Criterion 19.1.1.

19.3.4. *Let* $A \subset \mathrm{Gr}_n W$ *be an open subset such that the corresponding differential relation* $\mathcal{R}_A \subset J^1(V,W)$ *is ample. Let* $V \subset W$ *be an embedded manifold and* X *a tubular neighborhood of* $V \subset W$, *fibered over* V. *Then the differential relation* $\mathcal{R}_A^X \subset X^{(1)}$ *which defines the A-directed* **sections** *of the fibration* $p : X \to V$ *is open and ample.*

Now we can proceed in the following way. Chose a sequence of maps $\Phi^{(i)} = \Phi^{t_i}$, $0 = t_0 < t_1 < \cdots < t_N = 1$, such that the angle between $\Phi^{(i)}(L)$ and $\Phi^{(i+1)}(L)$ is less than, say, $\pi/4$ for all n-planes $L \in \mathrm{Gr}_n W$. Set $A_i = \Phi_*^{(i)} A$. Consider a tubular neighborhood X of the submanifold $f_0(V) \subset W$ and the differential relation $\mathcal{R}_{A_1}^X$ in $X^{(1)}$. This relation is open and ample and hence we can apply convex integration to its formal solution

$F = \Phi^{(1)}(df_0(V))$. Let $f_{t_1} : V \to X \subset W$ be the resulting embedding. Now we can apply the same construction to a tubular neighborhood X of $f_{t_1}(V) \subset W$, the differential relation $\mathcal{R}^X_{A_2} \subset X^{(1)}$ and its formal solution $F = \Phi^{(2)}(df_{t_1}(V))$. We can continue this way. The embedding $f_{t_N} = f_1$ will have the required properties. The approximation property follows from the possibility to approximate at each step. □

19.3.5. (Corollary: real embeddings, Gromov [Gr86]) *Let (W, J) be an almost complex manifold. Then*

- **(a)** *Every embedding $f_0 : V \to (W, J)$ whose tangential lift $G_0 = \mathrm{G}df_0 : V \to \mathrm{Gr}_n W$ is homotopic over embeddings to a map $G_1 : V \to A_{\mathrm{real}} \subset \mathrm{Gr}_n W$ (resp. $G_1 : V \to A_{\mathrm{coreal}} \subset \mathrm{Gr}_n W$) can be isotoped to a real (resp. co-real) embedding $f_1 : V \to W$.*
- **(b)** *Let $f_t : V \to (W, J)$, $t \in [0, 1]$, be an isotopy which connects two real (resp. co-real) embeddings f_0 and f_1. Suppose that there exists a family of real (resp. co-real) homomorphisms $F_t : TV \to TW$ which covers the isotopy f_t, $t \in [0, 1]$ and such that the families $df_t, F_t : TV \to TW$, $t \in [0, 1]$, are homotopic via families of monomorphisms fixed at $t = 0, 1$. Then there exists an isotopy of real (resp. co-real) embeddings $\widetilde{f_t} : V \to W$, $t \in [0, 1]$ which connects $\widetilde{f_0} = f_0$ with $\widetilde{f_1} = f_1$, is C^0-close to the isotopy f_t, and such that the families $d\widetilde{f_t}, F_t : TV \to TW$, $t \in [0, 1]$, are homotopic via families of monomorphisms fixed at $t = 0, 1$.*

◄ **Remark.** It is important to realize that just the existence of the family of real monomorphisms $F_t : TV \to TW$ which covers the isotopy f_t, $t \in [0, 1]$, is not sufficient for the existence of a real isotopy connecting f_0 and f_1, see [**Fd87**] and [**Plt88**]. Thus the existence of "homotopy of homotopies" is crucial and cannot be omitted. Algebro-topological consequences of this condition were computed in several examples by Borrelli (see [**Bo01**]). ►

Chapter 20

First Order Linear Differential Operators

All the examples below fit into the philosophy of Σ-non-singular solutions, see Section 5.2. The singularity $\Sigma \subset X^{(1)}$ in these examples will have the form $\Sigma = D^{-1}(S)$ where D is the symbol of a first order linear differential operator $\mathcal{D} : \operatorname{Sec} X \to \operatorname{Sec} Z$ and $S \subset Z$ is a subset of the total space of the vector bundle Z.

20.1. Formal inverse of a linear differential operator

Let X and Z be vector bundles over V. Note that in this case the fibration $X^{(1)} \to V$ has a natural linear structure. A first order linear differential operator

$$\mathcal{D} : \operatorname{Sec} X \to \operatorname{Sec} Z$$

can be written as a composition

$$\operatorname{Sec} X \xrightarrow{J^1} \operatorname{Sec} X^{(1)} \xrightarrow{\widetilde{D}} \operatorname{Sec} Z$$

where the map $\widetilde{D}$ is induced by a fiberwise homomorphism

$$X^{(1)} \xrightarrow{D} Z$$

of vector bundles over V. The vector bundle homomorphism $D = \operatorname{Symb} \mathcal{D}$ is called the *symbol* of the operator $\mathcal{D}$.

Suppose $D = \operatorname{Symb} \mathcal{D}$ is a fiberwise *epimorphism*. Then D can be viewed as an affine fibration $D : X^{(1)} \to Z$, and thus we have a homotopy equivalence $\operatorname{Sec} X^{(1)} \simeq \operatorname{Sec} Z$. In particular, any section $s : V \to Z$ can be lifted in a homotopically canonical way to a section $F_s : V \to X^{(1)}$ such that $D \circ F_s = s$. It is useful to think of F_s as a "formal inverse" of s. Thus we can say that the differential operator $\mathcal{D}$ with a surjective symbol is *formally invertible.* If in addition the operator $\mathcal{D}$ *depends only on derivatives of a section of* X*, and does not depend on the values of the section*[1], i.e. if the symbol D can be written as the composition

$$X^{(1)} \overset{\text{pr}}{\to} X^{(1)}/X \to Z\,,$$

then $\mathcal{D}$ is formally invertible *over every fixed section* $\gamma : V \to X$, i.e. the formal inverse F_s for $s : V \to Z$ can be chosen in a such way that $\operatorname{bs} F_s = \gamma$.

◀ **Example**: *Formal primitive of a differential form* (compare 4.7).
The symbol D of the the exterior differentiation

$$d : \operatorname{Sec} \Lambda^{p-1} V \to \operatorname{Sec} \Lambda^p V$$

is a fiberwise *epimorphism*

$$(\Lambda^{p-1} V)^{(1)} \to \Lambda^p V.$$

Therefore, d is formally invertible. Moreover, d is pure differential and hence it is formally invertible over any differential $(p-1)$-form. ▶

20.2. Homotopy principle for $\mathcal{D}$-sections

Let $\mathcal{D} : \operatorname{Sec} X \to \operatorname{Sec} Z$ be a differential operator. A section $s : V \to Z$ is called $\mathcal{D}$-section if $s = \mathcal{D} f$ for a section $f : V \to X$. For example if $\mathcal{D}$ is the exterior differentiation

$$d : \operatorname{Sec} \Lambda^{p-1} V \to \operatorname{Sec} \Lambda^p V\,,$$

then the $\mathcal{D}$-sections are *exact* differential p-forms.

Given a subset $S \subset Z$ we will denote by $\operatorname{Sec}_{\mathcal{D}}(Z \setminus S)$ the space of all $\mathcal{D}$-sections $V \to Z \setminus S$.

Let us point out the following important but trivial h-principle.

20.2.1. (Homotopy principle for $\mathcal{D}$-sections) *Let* $\mathcal{D} : \operatorname{Sec} X \to \operatorname{Sec} Z$ *be a linear differential operator such that* $D = \operatorname{Symb} \mathcal{D}$ *is a fiberwise epimorphism. Let* S *be a subset of* Z*. If the* h*-principle holds for* Σ*-non-singular sections* $V \to X$*, where* $\Sigma = D^{(-1)}(S) \subset X^{(1)}$*, then it also holds for the inclusion*

$$\operatorname{Sec}_{\mathcal{D}}(Z \setminus S) \hookrightarrow \operatorname{Sec}(Z \setminus S)\,,$$

i.e. for any section $s_0 \in \operatorname{Sec}(Z \setminus S)$ *there exists a homotopy* $s_t : I \to \operatorname{Sec}(Z \setminus S)$ *such that* $s_1 \in \operatorname{Sec}_{\mathcal{D}}(Z \setminus S)$*. The same is also true for all forms*

[1]Operators of this type we will further call *pure differential.*

of the h-principle, except the C^0-dense one (which is not defined for the inclusion $\mathrm{Sec}_{\mathcal{D}}(Z \setminus S) \hookrightarrow \mathrm{Sec}\,(Z \setminus S)$ *).*

If the operator $\mathcal{D}$ is pure differential, then the C^0-dense h-principle for Σ-non-singular sections $V \to X$ implies that for any section $f_0 : V \to X$ one can choose the homotopy s_t in such a way that $s_1 = \mathcal{D}f_1$, where f_1 is arbitrarily C^0-close to f_0 (and similarly for all other forms of the C^0-dense h-principle).

The proof follows immediately from the homotopy equivalence

$$\mathrm{Sec}\,(X^{(1)} \setminus \Sigma) \simeq \mathrm{Sec}\,(Z \setminus S)\,.$$

The C^0-dense version follows from existence of a formal inversion over any fixed section $f : V \to X$.

20.3. Non-vanishing $\mathcal{D}$-sections

A section $s : V \to Z$ is called *non-vanishing* if $s(v) \neq 0$ for all $v \in V$.

We say that a linear differential operator $\mathcal{D}$ has *principal* rank ≥ 2 if $\dim D(P) \geq 2$ for each principal subspace P in $X^{(1)}$. The following theorem was first proved in [**GE71**] using the method of removal of singularities.

20.3.1. (Homotopy principle for non-vanishing $\mathcal{D}$-sections) *If the linear differential operator $\mathcal{D}$ has principal rank ≥ 2 and $D = \mathrm{Symb}\,\mathcal{D}$ is a fiberwise epimorphism then all the forms of the h-principle (excluding the C^0-dense one) hold for the inclusion*

$$\mathrm{Sec}_{\mathcal{D}}(Z \setminus \mathbf{0}) \hookrightarrow \mathrm{Sec}\,(Z \setminus \mathbf{0})\,,$$

where $\mathbf{0} \subset Z$ is the zero-section. In particular, any non-vanishing section $s : V \to Z$ can be deformed via a homotopy of non-vanishing sections to a section $\mathcal{D}f$. Moreover, if $\mathcal{D}$ is pure differential then we can choose f arbitrarily C^0-close to any fixed section $f_0 : V \to X$.

Proof. According to Proposition 20.2.1 it is sufficient to prove the h-principle for Σ-non-singular sections $V \to X$, where $\Sigma = D^{-1}(\mathbf{0}) = \mathrm{Ker}\,D$. The inequality $\dim D(P) \geq 2$ is equivalent to the inequality

$$\mathrm{codim}_P(P \cap \mathrm{Ker}\,D) \geq 2$$

and hence the singularity Σ is thin. Therefore we can apply Theorem 18.4.2. □

20.3.2. (Corollary) *Let $n \geq 3$ and $2 \leq p \leq n-1$. Any non-vanishing differential p-form on an n-dimensional manifold V can be deformed via a homotopy of non-vanishing forms to a non-vanishing exact form.*

Proof. Let us check that the inequalities $n \geq 3$ and $2 \leq p \leq n-1$ imply that the principal rank of the exterior differentiation d is ≥ 2. Let $X = \Lambda^{p-1}V$, $Z = \Lambda^p V$ and D be the symbol of d. Let P be a coordinate principal subspace which corresponds, say to the first coordinate x_1 of a local coordinate system $x_1, \dots, x_n$ on V. The dimension of P is equal to $C_n^{p-1} = \frac{n!}{(p-1)!(n-p+1)!}$. If the intersection $(\operatorname{Ker} D) \cap P$ is not empty, then in local coordinates it is defined by the system of C_{n-1}^{p-1} equations

$$a^1_{i_1 \dots i_{p-1}} = \text{const}\,, \ 1 \notin \{i_1 \dots i_{p-1}\}\,,$$

where the coordinates $\{a^1_{i_1 \dots i_{p-1}}\}$ correspond to the derivatives $\partial/\partial x_1$ of the coefficients of $(p-1)$-forms. Hence for $2 \leq p \leq n-1$ and $n \geq 3$ we have

$$\operatorname{rank} D = C_{n-1}^{p-1} \geq 2\,,$$

and therefore Theorem 20.3.1 applies. □

20.3.3. (Corollary) *Let $n \geq 3$ and V be endowed with a volume form Ω. Then any non-vanishing vector field L on V is homotopic through non-vanishing vector fields to a divergence free vector field.*

Proof. By Cartan's formula we have

$$\mathcal{L}_L \Omega = d(L \lrcorner\, \Omega) + L \lrcorner\, d\Omega = d(L \lrcorner\, \Omega)\,.$$

Therefore the flow of the field L preserves Ω if and only if the $(n-1)$-form $L \lrcorner\, \Omega$ is closed. The correspondence $\mathbf{v} \mapsto \omega_{\mathbf{v}} = \mathbf{v} \lrcorner\, \Omega$ is a fiberwise isomorphism $TV \to \Lambda^{n-1}V$. Thus Theorem 20.3.2 implies that ω_L is homotopic through non-vanishing forms to an exact form $\omega = \omega_{L_1}$, and then L_1 will be a divergence free vector field. □

◀ **Exercise.** Let $n \geq 3$, $2 \leq p \leq n-1$ and $a \in H^p(V)$. Prove that any non-vanishing differential p-form on an n-dimensional manifold V can be deformed via a homotopy of non-vanishing forms to a *closed* form which represents the class a. *Hint*: consider the singularity $\Sigma = D^{-1}(-\omega_a)$, where ω_a is a closed p-form which represents a. ▶

20.4. Systems of linearly independent $\mathcal{D}$-sections

In all applications of convex integration which we considered so far the relation $\mathcal{R}$ was a complement of a thin singularity. In this chapter we will consider an example when the singularity

$$\Sigma = D^{-1}(S) \subset X^{(1)}$$

is not thin.

The sections of a vector bundle are called linearly independent if they are *pointwise* linearly independent. Gromov proved in [**Gr86**] the following

20.4.1. (Homotopy principle for systems of linearly-independent $\mathcal{D}$-sections) *Let $\mathcal{D} : \operatorname{Sec} X \to \operatorname{Sec} Z$ be a differential operator of principal rank ≥ 2 such that its symbol $D = \operatorname{Symb} \mathcal{D}$ is a fiberwise epimorphism. Then any system $\{s_i\} = \{s_1, \dots, s_k\}$ of linearly independent sections of the vector bundle Z can be deformed via a homotopy of systems of linearly independent sections to a system of sections $\{\mathcal{D} f_i\}$.*

Proof. It is sufficient to consider the case when Z is a trivial bundle and $\{s_i\}$ is a trivialization of Z. Write

$$\overline{X} = \underbrace{X \oplus \cdots \oplus X}_{q}, \quad \overline{Z} = \underbrace{Z \oplus \cdots \oplus Z}_{q} \quad \text{and} \quad \overline{\mathcal{D}} = \underbrace{\mathcal{D} \oplus \cdots \oplus \mathcal{D}}_{q}.$$

Let $\Sigma = \overline{D}^{-1}(S) \subset \overline{X}^{(1)}$, where $\overline{D}$ is the symbol of $\overline{\mathcal{D}}$, and $S \subset \overline{Z}$ is given in the fibers of the fibration $\overline{Z} \to V$ by the equation $z_1 \wedge \cdots \wedge z_q = 0$. Then the singularity $\Sigma \subset X^{(1)}$ is defined in the fibers $\overline{L}_v = L_v \oplus \cdots \oplus L_v$ of the fibration $\overline{X}^{(1)} \to V$ by the equation

$$D(y^1) \wedge \cdots \wedge D(y^q) = 0,$$

where $y^1, \dots, y^q \in L_v$.

According to Proposition 20.2.1 we just need to check that the differential relation $\mathcal{R} = \overline{X}^{(1)} \setminus \Sigma$ is ample. A principal subspace $\overline{P}$ in $\overline{X}^{(1)}$ over a point $v \in V$ is the Cartesian product of q principal subspaces $P_1, \dots, P_q$ in $X^{(1)}$ over v. These spaces are *parallel* affine subspaces in the fiber L_v of the bundle $X^{(1)} \to V$. Therefore the images $P'_1 = D(P_1), \dots, P'_q = D(P_q)$ are parallel affine equidimensional subspaces in the fiber Z_v of the bundle Z. The operator $\mathcal{D}$ has principal rank ≥ 2 and hence $r = \dim P'_1 = \cdots = \dim P'_q \geq 2$. To simplify the notation, we will further assume that $r = 2$; the case $r > 2$ can be considered in a similar way.

Choose a basis $w_1, \dots w_q$ in Z_v such that w_1 and w_2 are parallel to P'_i. For each $i = 1, \dots, q$ set $a^i = P'_i \cap \operatorname{Span}\{w_3, \dots, w_q\}$. The coordinates of a^i in $w_1, \dots, w_q$ are $(0, 0, a^i_3, \dots, a^i_q)$.

For each $i = 1, \dots, q$ fix an origin 0 in the affine subspace P_i such that $0 \in D^{-1}(a^i)$, and choose a basis $v^i_1, \dots, v^i_q$ in $(P_i, 0)$ such that

$$D(v^i_1) = a^i + w_1, \quad D(v^i_2) = a^i + w_2,$$

and the vectors $v^i_3, \dots, v^i_q$ belong to the kernel of the map $D|_{P_i} : (P_i, 0) \to (P'_i, a^i)$. Let $\{\alpha^i_j\}$ and $\{\beta_j\}$ be the coordinates in $\overline{P}$ and Z_v which correspond to the basis $\{v^i_j\}$ in $\overline{P}$ and to the basis $\{w_j\}$ in Z_v. In these coordinates the non-empty intersection $\mathcal{R} \cap \overline{P}$ is given by the non-equality

$$\det A = \begin{vmatrix} \alpha_1^1 & \alpha_1^2 & \dots & \dots & \alpha_1^q \\ \alpha_2^1 & \alpha_2^2 & \dots & \dots & \alpha_2^q \\ a_3^1 & a_3^2 & \dots & \dots & a_3^q \\ \dots & \dots & \dots & \dots & \dots \\ a_q^1 & a_q^2 & \dots & \dots & a_q^q \end{vmatrix} \neq 0 \,,$$

where a_j^i are *constants*, i.e. they do not depend on the coordinates $\{\alpha_l^k\}$. Therefore,

$$\overline{\mathcal{R}} \cap \overline{P} = (\mathbb{R}^{2q} \setminus \{\det A = 0\}) \times \mathbb{R}^{q(q-2)} \,.$$

The complement $\mathbb{R}^{2q} \setminus \{\det A = 0\}$ consists of two (non-empty) path connected components: a positive one where $\det A > 0$, and a negative one where $\det A < 0$. We need to prove that the convex hull of each of these components coincides with $\mathbb{R}^{2q}$. Let $\{A_j^i\}_{j=1,2}^{i=1,\dots,q}$ be the matrices which correspond to the standard basis in $\mathbb{R}^{2q}$. For every matrix A_0 there exists a constant a such that A_0 belongs to the interior of the convex hull of $4q$ matrices $\{\pm a \cdot A_j^i\}$. Each of the matrices $\pm a \cdot A_j^i$ belongs to the quadric $\{\det A = 0\}$. These matrices can be slightly moved into, say the positive component so they would still contain A_0 in the interior of their convex hull. □

20.4.2. (Corollary: systems of exact forms) *Let $\{\omega_i\}_{i=1,\dots,q}$ be a system of linearly independent differential p-forms on V. If $2 \le p \le n-1$, where $n = \dim V$, then $\{\omega_i\}$ can be deformed via a homotopy of systems of linearly independent form to a system of exact linearly independent forms.*

20.4.3. (Corollary: system of divergence free vector fields) *Let $n = \dim V \ge 3$ and V be endowed with a volume form Ω. Any system of linearly independent vector fields on V can be deformed via a homotopy of systems of linearly independent vector fields to a system of divergence free vector fields. In particular, every parallelizable manifold supports $n = \dim V$ linearly independent divergence free vector fields.*

20.5. Two-forms of maximal rank on odd-dimensional manifolds

As we already had seen in Chapter 10.4 above, Gromov's h-principle for symplectic forms on open manifolds implies McDuff's h-principle for closed 2-forms of maximal rank on closed odd-dimensional manifolds. McDuff proved in [**MD87a**] this h-principle using the convex integration technique by showing that the corresponding differential relation is ample. We reproduce her argument in this chapter.

Let V be a manifold of dimension $n = 2m + 1$. For a fixed 2-form ω_0 on V we define $S_{\omega_0} \subset \Lambda^2 V$ by the equation $(z + \omega_0(v))^m = 0$ for each $v \in V$. Let D be the symbol of the exterior differentiation $d : \operatorname{Sec} \Lambda^1 V \to \operatorname{Sec} \Lambda^2 V$.

20.5.1. (McDuff, [**MD87a**]**)** *For any differential 2-form ω_0 on V the differential relation $\mathcal{R} = (\Lambda^1 V)^{(1)} \setminus \Sigma$, where $\Sigma = D^{-1}(S_{\omega_0})$, is ample.*

According to Theorem 20.2.1 the ampleness of $\mathcal{R}$ implies that

all forms of the h-principle (excluding the C^0-dense one) hold for the inclusion

$$\operatorname{Sec}_d (\Lambda^2 V \setminus S_{\omega_0}) \hookrightarrow \operatorname{Sec} (\Lambda^2 V \setminus S_{\omega_0}),$$

where $\operatorname{Sec}_d (\Lambda^2 V \setminus S_{\omega_0})$ is the space of exact sections $V \to \Lambda^2 V \setminus S_{\omega_0}$.

This is equivalent to the h-principle 10.4.1. In particular, if V supports a 2-form of maximal rank then every two-dimensional cohomology class of V can be represented by a (closed) non-degenerate form.

Proof of Theorem 20.5.1.

To simplify notation we assume that $\omega_0 = 0$. The proof can be easily rewritten for any ω_0. In local coordinates the singularity Σ is defined by the equation

$$\Omega = \Sigma \alpha_i l_i = [\Sigma_{i<j} (y_{ij} - y_{ji}) dx_i \wedge dx_j]^m = 0,$$

where the coordinates $\{y_{ij}\}$ correspond to the derivatives $\partial a_j / \partial x_i$ of the coefficients of the 1-forms $\Sigma a_j dx_j$ and l_i is the exterior product of all basic 1-forms dx_j, $j = 1, \dots, n$, excluding dx_i. In a coordinate principal subspace P which corresponds, say, to the first coordinate in V, only the y_{1j} are *variables*; all the other $y_{ij} = y^0_{ij}$ are *constants*. In particular, the coefficient α_1 is constant. If $\alpha_1 \neq 0$ then $\Sigma \cap P$ is empty. Otherwise $\Sigma \cap P$ is defined by the system of $(n-1)$ linear equations $\{\alpha_i = 0\}_{i=2,\dots,n}$, which can be written as $AZ = 0$, where A is a constant matrix, $Z = (z_2, \dots, z_n)$ and $z_j = y_{1j} - y^0_{j1}$. The condition $\Sigma \neq P$ implies $A \neq 0$. It is easy to check that $a_{ii} = 0$ and $a_{ij} = \pm a_{ji}$. Hence, the rank of A is at least 2 and thus the singularity Σ is thin. Therefore $\mathcal{R}$ is ample. □

20.6. One-forms of maximal rank on even-dimensional manifolds

Let $V = V^{2m}$ be an even-dimensional manifold. As we already have shown above in Chapter 14 McDuff's h-principle 14.2.3 for maximally non-integrable hyperplane distributions cab be deduced from two of Gromov's h-principles: 10.3.2 for contact structures on open manifolds, and 14.2.1 for mappings of closed manifolds transversal to a contact structure. In this section we reconstruct McDuff's original argument based on the convex integration technique.

Let us recall that a 1-form α on V is called *maximally non-degenerate* if the differential $(2m-1)$-form $\alpha \wedge (d\alpha)^{m-1}$ never vanishes. A pair of differential forms (α, β) on V, where α is a 1-form and β is a 2-form, is called *maximally non-degenerate* if the differential $(2m-1)$-form $\alpha \wedge \beta^{m-1}$ never vanishes. A pair (α, β) is called *exact* if $\beta = d\alpha$.

Let the linear differential operator

$$\mathcal{D} = (\mathrm{id}, d) : \mathrm{Sec}\, \Lambda^1 V \to \mathrm{Sec}\, (\Lambda^1 V \oplus \Lambda^2 V)$$

be defined by the formula $\alpha \mapsto (\alpha, d\alpha)$. Then the $\mathcal{D}$-sections

$$V \to \Lambda^1 V \oplus \Lambda^2 V \setminus S_{\omega_0}$$

are exact pairs $(\alpha, d\alpha)$. Note that the symbol

$$D : (\Lambda^1 V)^{(1)} \to \Lambda^1 V \oplus \Lambda^2 V$$

of the operator $\mathcal{D} = (\mathrm{id}, d)$ is fiberwise *epimorphic*.

For an even-dimensional manifold $V = V^{2m}$ let a subset $S \subset \Lambda^1 V \oplus \Lambda^2 V$ be defined in the fibers of the fibration $\Lambda^1 V \oplus \Lambda^2 V \to V$ by the equation

$$z_1 \wedge (z_2)^{m-1} = 0.$$

20.6.1. (McDuff, [MD87a]) *The differential relation* $\mathcal{R} = (\Lambda^1 V)^{(1)} \setminus \Sigma$, *where* $\Sigma = D^{-1}(S)$, *is ample.*

According to Theorem 20.2.1 the ampleness of $\mathcal{R}$ implies that

all forms of the h-principle (excluding the C^0*-dense one) hold for the inclusion*

$$\mathrm{Sec}_{\mathcal{D}}\, (\Lambda^1 V \oplus \Lambda^2 V \setminus S) \hookrightarrow \mathrm{Sec}\, (\Lambda^1 V \oplus \Lambda^2 V \setminus S)\,,$$

where $\mathrm{Sec}_{\mathcal{D}}\, (\Lambda^1 V \oplus \Lambda^2 V \setminus S)$ *is the space of* $\mathcal{D}$*-sections* $V \to \Lambda^1 V \oplus \Lambda^2 V \setminus S_{\omega_0}$.

This is equivalent to the h-principle 14.2.3, and in particular, every non-degenerate pair of forms (α, β) on V can be deformed via a homotopy of non-degenerate pairs to an exact non-degenerate pair $(\widetilde{\alpha}, d\widetilde{\alpha})$. The C^0-dense h-principle holds in the following version: one can choose $\widetilde{\alpha}$ to be arbitrarily C^0-close to α (and similarly for all other forms of the C^0-dense h-principle).

Proof of Theorem 20.6.1.

As in the proof of Theorem 20.5.1 the key observation is that the matrix of a system of linear equations which defines the intersection of a principal subspace with Σ is *almost skew-symmetric*: $a_{ij} = \pm a_{ji}$ and $a_{ii} = 0$. Hence its rank cannot be equal to 1, and thus the corresponding singularity Σ is thin. To clarify the computation we consider only the case $m = 2$. The general case can be treated in a similar way (see [**MD87a**]).

Let the coordinates $\{y_{ij}\}$ correspond to the derivatives $\partial a_j/\partial x_i$ of the coefficients of the 1-forms $\Sigma a_j dx_j$ and $z_{ij} = y_{ij} - y_{ji}$, $i < j$. For $m = 2$ the singularity Σ is defined in local coordinates by the equation $\omega_1 \wedge \omega_2 = 0$, where

$$\begin{aligned}\omega_1 &= a_1 dx_1 + a_2 dx_2 + a_3 dx_3 + a_4 dx_4\,,\\ \omega_2 &= z_{12} dx_1 \wedge dx_2 + z_{13} dx_1 \wedge dx_3 + z_{14} dx_1 \wedge dx_4 + \\ &\quad + z_{23} dx_2 \wedge dx_3 + z_{24} dx_2 \wedge dx_4 + z_{34} dx_3 \wedge dx_4\,.\end{aligned}$$

This equation is equivalent to the system of equations

$$\begin{array}{ccccccc} a_4 z_{23} & - & a_3 z_{24} & + & a_2 z_{34} & = & 0 \\ 0 & + & a_4 z_{13} & - & a_3 z_{14} & = & -a_1 z_{34} \\ a_4 z_{12} & + & 0 & - & a_2 z_{14} & = & -a_1 z_{24} \\ a_3 z_{12} & - & a_2 z_{13} & + & 0 & = & -a_1 z_{23} \end{array}$$

In a principal subspace P which corresponds, say, to the first coordinate in V, only the z_{1j} are *variables*; all the other z_{ij}, and also a_i, are *constants.* In particular, the first equation does not contain unknowns and hence $\Sigma \cap P = \varnothing$ if this equation is not an identity. If $\Sigma \neq P$ then the system of the last three equations, which contain three unknowns z_{12}, z_{13}, z_{14}, has rank ≥ 2 and hence the singularity Σ is thin. □

Chapter 21

Nash-Kuiper Theorem

21.1. Isometric immersions and short immersions

Recall that a C^r-smooth family $g = \{g_x\,,\ x \in V\}$ of positive quadratic forms on T_xV, $x \in V$, is called a *Riemannian C^r-metric* on V, $r = 0, 1, \ldots$. The pair (V, g) is then called a *Riemannian C^r-manifold*. In what follows the class of the Riemannian metric is not essential and we will write *Riemannian manifold* instead of *Riemannian C^r-manifold* and so on.

We will consider also, as a technical tool, families of *non-negative* quadratic forms on V. Such a family will be called a *semi-Riemannian metric* on V.

Let (V^n, g) and (W^q, h) be Riemannian manifolds. A C^1-smooth map $f : V \to W$ is called *isometric* if $f^*h = g$, i.e. $d_xf : T_xV \to f_x(T_xV) \subset T_{f(x)}W$ is a linear isometry for every $x \in V$. Any isometric map is automatically an immersion. Locally with respect to a frame of independent vector fields $\{\partial_i\}$, $i = 1, \ldots, n$, the isometry condition can be described by the system of equations

$$\langle f_*\partial_i, f_*\partial_j\rangle_h = \langle \partial_i, \partial_j\rangle_g\,,\ 1 \le i \le j \le n\,,$$

where $f_*\partial_i = df(\partial_i) = \partial_i f$. Note that this system is overdetermined when $q < \frac{n(n+1)}{2}$.

A C^1-map $f : V \to W$ is called *strictly short* if

$$f^*h < g\,,\ \text{ i.e. } ||\,f_*v||_h < ||\,v||_g$$

for all tangent vectors $v \in TV$. A C^1-map $f : V \to W$ is called *short* if $f^*h \le g$. A (strictly) short map is not necessarily an immersion.

◀ **Example.** Given an arbitrarily C^1-map $f : (V, g) \to \mathbb{R}^q$, the composition $H_a \circ f : (V, g) \to \mathbb{R}^q$, where $H_a(x) = ax$ is a homothety centered at the origin, is strictly short for all sufficiently small $a > 0$. ▶

21.2. Nash-Kuiper theorem

It is well known from classical differential geometry that for $r > 1$ the C^r-smooth isometric immersions of two-dimensional Riemannian C^∞-manifolds into $\mathbb{R}^3$ are very specific and rigid maps. For example, any isometric C^2-immersions of the standard sphere $S^2 \subset \mathbb{R}^3$ into $\mathbb{R}^3$ is congruent to the standard embedding $S^2 \hookrightarrow \mathbb{R}^3$. Till the middle of 1950's mathematicians mostly believed that C^1-smooth isometric immersions $V^n \to W^q$ are also rigid and hard to construct, and, in particular, the aforementioned uniqueness survives also for isometric immersions $S^2 \to \mathbb{R}^3$ which are only C^1-smooth.

It was discovered by J. Nash in 1954 that the situation is, in fact, drastically different when one passes to C^1-smooth immersions. On contrast with C^2-immersions they appeared to be extremely flexible:

21.2.1. (Nash-Kuiper) *If $n < q$ then any strictly short immersion*

$$f : (V^n, g) \to (\mathbb{R}^q, h)\,,$$

where h is the standard metric on $\mathbb{R}^q$, can be C^0-approximated by isometric C^1-smooth immersions. Moreover, if the initial immersion f is an embedding then f can be approximated by isometric C^1-embeddings.

For example there exists a C^1-isometric embedding of the standard sphere S^2 and the standard disk D^2 into an arbitrarily small ball in $\mathbb{R}^3$.

Nash proved in [**Na54**] this theorem for $n \leq q - 2$ and later Kuiper in [**Ku55**] extended the theorem to the case $n = q - 1$. The parametric version of the theorem is also true and implies (together with the **Example** in 21.1) the following

21.2.2. *Isometric C^1-immersions $V^n \to \mathbb{R}^q$, $n < q$, satisfy the parametric h-principle for all Riemannian manifolds $V = (V, g)$.*

◀ **Remark.** The C^0-dense h-principle will also hold if the shortness condition is incorporated in the definition of a formal isometric immersion. ▶

The rest of this chapter is devoted to the proof of Theorem 21.2.1. We will consider only the case when V is compact and f is an embedding. The case of immersions follows formally from the case of embeddings. The proof can be easily adjusted for non-compact manifolds and also for the parametric case. Moreover, it can be generalized for a general target manifold (W, h) without employing any additional ideas.

21.3. Decomposition of a metric into a sum of primitive metrics

A quadratic form Q is called *primitive* if $Q = l^2$ where l is a linear form. A semi-Riemannian metric g on $\mathbb{R}^n$ is called *primitive* if $g = \alpha(x)(dl)^2$, where $l = l(x)$ is a linear function on $\mathbb{R}^n$ and α is a non-negative function with compact support.

A semi-Riemannian metric g on a manifold V is called *primitive* if there exists a local parametrization $u : \mathbb{R}^n \to U \subset V$ such that $\operatorname{supp} g \subset U$ and u^*g is a primitive metric on $\mathbb{R}^n$.

21.3.1. (Lemma) *Any Riemannian metric g on a compact manifold V can be decomposed into a finite sum of primitive metrics.*

Proof. Choose a set of local parametrizations $\{u_i : \mathbb{R}^n \to U_i \subset V\}$ and a partition of unity $\{\alpha_i\}$, $\sum \alpha_i \equiv 1$, on V such that $\operatorname{supp} \alpha_i \subset U_i$.

Let $A_i = \operatorname{supp}(\alpha_i \circ u_i)$ and $\widetilde{g}^i = (u_i^* g)|_{A_i}$. For every i we can find positive quadratic forms Q_{ij}, $j = 1, \ldots, N(i)$, on $\mathbb{R}^n$ such that for every $x \in A_i$ the positive quadratic form $\widetilde{g}^i_x = \widetilde{g}^i|_{T_x\mathbb{R}^n = \mathbb{R}^n}$ belongs to the interior of the convex hull of the forms Q_{ij}, $j = 1, \ldots, N(i)$. Every positive quadratic form Q_{ij} is a sum of some primitive forms $(l_{ijk})^2$, $k = 1, \ldots, n$. Therefore,

$$g = \sum_{ijk} \alpha_i g_{ijk} \quad \text{where} \quad g_{ijk} = (u_i^{-1})^*(dl_{ijk})^2$$

is the desired decomposition. $\square$

◀ **Remark.** For any positive function β on V we can decompose a metric $\widetilde{g}$ which is sufficiently C^0-close to $\beta(x)g$ into a sum of primitive metrics using *the same set* of forms g_{ijk}. ▶

21.4. Approximation theorem

A. The functions $r(\widetilde{g}, g)$ and $d_g(\widetilde{f}, f)$

Given a pair of metrics g and $\widetilde{g}$ on V we will denote by $r(\widetilde{g}, g)$ the function

$$TV \setminus V \to \mathbb{R}, \quad v \to r(\widetilde{g}, g)(v) = \frac{||\,v||_{\widetilde{g}}}{||\,v||_g}\,.$$

The function $r(\widetilde{g}, g)$ is defined also in the case when $\widetilde{g}$ is a semi-Riemannian metric on V. Note that

(r1) $r(\widetilde{g}, g_1) \le r(\widetilde{g}, g_2)$ if $g_1 \ge g_2$ and

(r2) $r(\widetilde{g}, g) \le r(\widetilde{g} + g_1, g + g_1)$ if $r(\widetilde{g}, g) \le 1$.

Given a pair of maps $f, \widetilde{f} : (V, g) \to \mathbb{R}^q$, we will denote by $d_g(f, \widetilde{f})$ the function

$$TV \setminus V \to \mathbb{R}\,, \quad v \to d_g(f, \widetilde{f})(v) = \frac{||\, f_* v - \widetilde{f}_* v||_h}{||\, v||_g}\,,$$

where h is the standard metric on $\mathbb{R}^q$. Note that

(d1) $d_{g_1}(f, \widetilde{f}) \le d_{g_2}(f, \widetilde{f})$ if $g_1 \ge g_2$.

The functions $r(\widetilde{g}, g)$ and $d_g(\widetilde{f}, f)$ do not depend on the lengths of v and in what follows we will consider the restriction of these functions to the g-unit tangent bundle $T_1 V$.

We will need the following lemma

21.4.1. (Convergence Lemma) *Let $f_i : V \to \mathbb{R}^q$ be a sequence of (smooth) maps. If $f_i \overset{C^0}{\to} \overline{f}$ and $d_g(f_i, f_{i+1}) < c_i$ with $\sum_i c_i < \infty$, then $\overline{f}$ is a C^1-smooth map and*

$$f_i \overset{C^1}{\to} \overline{f}\,.$$

Indeed, the convergence of the series $\sum_{i=1}^{\infty} d_g(f_i, f_{i+1})$ is just the Cauchy condition for first derivatives of the sequence f_i.

B. Approximation theorem

An embedding $f : (V, g) \to (\mathbb{R}^q, h)$ is called *ε-isometric* if

$$(1 - \varepsilon) g < f^* h < (1 + \varepsilon) g\,.$$

Instead of the theorem about C^1-isometric embeddings we will first prove the following

21.4.2. (Approximation Theorem) *Let $n < q$. For any $\varepsilon > 0$, any short embedding $f : (V^n, g) \to (\mathbb{R}^q, h)$ can be C^0-approximated by ε-isometric embeddings. Moreover, we can also control the C^1-closeness in the following sense: given a fixed decomposition of the semi-Riemannian metric $\Delta = g - f^* h$ into a sum of N primitive metrics then for any constant $\rho > 0$ we can choose an approximating embedding $\widetilde{f}$ which satisfies the inequality*

$$d_g(f, \widetilde{f}) \; < \; N\, r(\Delta, g) + \rho\,.$$

Our proof of the Nash-Kuiper theorem will consist of two parts. First we will use the first part of the Approximation theorem 21.4.2 to construct a sequence of embeddings f_i such that

$$f_i \overset{C^0}{\to} \overline{f} \qquad \text{and} \qquad f_i^* h \overset{C^0}{\to} g\,.$$

Figure 21.1. The relation $\mathcal{R}_f$ and $\widetilde{\mathcal{R}}_f$

Then we will refine our choice of the sequence f_i in order to have $\sum_i d_g(f_i, f_{i+1}) < \infty$. Then according to the Convergence Lemma 21.4.1 $\overline{f}$ will be a C^1-limit and $\overline{f}^* h = g$.

The next three sections are devoted to the proof of the Approximation Theorem 21.4.2.

21.5. One-dimensional approximation theorem

21.5.1. *For any $\varepsilon > 0$ any short embedding $f : (I, g) \to (\mathbb{R}^q, h)$ can be C^0-approximated by ε-isometric embeddings. Moreover, for any $\rho > 0$ the approximating map $\widetilde{f}$ can be chosen in such a way that*

$$d_g(f, \widetilde{f}) < r(\Delta, g) + \rho$$

where $\Delta = g - f^* h$.

Proof. Let $f : (I, g) \to (\mathbb{R}^q, h)$ be a strictly short embedding. Let τ be the orienting g-unit vector field on I, i.e. $\partial_\tau t > 0$ and $||\,\tau||_g = 1$.

The isometry condition $\mathcal{R}_{\mathrm{iso}} \subset J^1(I, \mathbb{R}^q) = I \times \mathbb{R}^q \times \mathbb{R}^q$ over a point $(t, y) \in I \times \mathbb{R}^q$ is the unit sphere $\Omega(t, y) = \{w \in \mathbb{R}^q\,,\ ||\,w||_h = 1\}$. Choose a normal vector field $\mathbf{n}$ to $f(I) \subset \mathbb{R}^q$. Instead of the relation $\mathcal{R}_{\mathrm{iso}}$ over $I \times \mathbb{R}^q$ we consider a smaller relation $\mathcal{R}_f \subset \mathcal{R}_{\mathrm{iso}}$ over $f(I) \subset I \times \mathbb{R}^q$, which consists of vectors $w \in \Omega(t, y)$ such that

$$w \in \mathrm{Span}\,\{f_*\tau,\, \mathbf{n}\} \ \text{ and } \ \langle w, f_*\tau\rangle_h \geq ||\, f_*\tau||_h^2$$

(see Fig.21.1). The pair $(f, f_*\tau/||\, f_*\tau||_h)$ is a short formal solution of $\mathcal{R}_f$. Let $\widetilde{\mathcal{R}}_f \subset J^1(I, \mathbb{R}^q)$ be a small open neighborhood of $\mathcal{R}_f \subset J^1(I, \mathbb{R}^q)$. Applying one-dimensional convex integration (Lemma 17.3.1) one can construct a solution $\widetilde{f}$ of $\widetilde{\mathcal{R}}_f$ which is arbitrarily C^0-close to f.

Figure 21.2. The foliations $\mathcal{P}$ and $\mathcal{L}$

If the map $\widetilde{f}$ is sufficiently C^0-close to f, then $\widetilde{f}$ will also be an embedding, because the angle between $f_*\tau$ and $\widetilde{f}_*\tau$ is less then $\frac{\pi}{2}$ − const.

For any $\rho > 0$ we can choose $\widetilde{\mathcal{R}}_f$ and $\widetilde{f}$ such that

$$d_g(f, \widetilde{f}) \,<\, r(\Delta, g) + \rho\,.$$

Indeed, using the Pythagorean theorem (see Fig.21.1) we have

$$d_g(f, \widetilde{f})(\tau) = ||\,\widetilde{f}_*\tau - f_*\tau||_h \,<\, \sqrt{1 - ||\,f_*\tau||_h^2} + \rho\,,$$

where $\rho \to 0$ if $\widetilde{\mathcal{R}}_f \to \mathcal{R}_f$. On the other hand,

$$\sqrt{1 - ||\,f_*\tau||_h^2} = \sqrt{(g - f^*h)(\tau)} = ||\,\tau||_{g-f^*h} = r(\Delta, g)(\tau)\,.$$

21.6. Adding a primitive metric

21.6.1. *Suppose that $n < q$. Let $f : (V, g) \to (\mathbb{R}^q, h)$ be a short embedding such that $\Delta = g - f^*h$ is a primitive metric on V. Then for any ε the embedding f can be C^0-approximated by ε-isometric embeddings. Moreover, for any $\rho > 0$ the approximating map $\widetilde{f}$ can be chosen to satisfy the inequality*

$$d_g(f, \widetilde{f}) \,<\, r(\Delta, g) + \rho\,.$$

Proof. It is sufficient to consider the case $(V, g) = (\mathbb{R}^n, g)$.

We are going to reduce this version of the approximation theorem to the parametric one-dimensional convex integration lemma (see 17.5.1). Let $g - f^*h = \alpha(x)(dl)^2$. The map f is isometric on each leaf of the $(n-1)$-dimensional affine foliation $\mathcal{P} = \{l(x) = \text{const}\}$. Let $\mathbf{v}$ be the vector field on $\mathbb{R}^n$ normal with respect to the metric g to the leaves of $\mathcal{P}$. Integral trajectories of $\mathbf{v}$ form a one-dimensional foliation $\mathcal{L}$ normal (with respect to the metric g) to the foliation $\mathcal{P}$ (see Fig.21.2).

We can choose a *global* frame ∂_i, $i = 1, \dots, n$, on $V = \mathbb{R}^n$ such that ∂_1 is tangent to $\mathcal{L}$ and ∂_i , $i = 2, \dots, n$, are tangent to $\mathcal{P}$. Therefore

$$\begin{cases} \langle f_*\partial_i, f_*\partial_j\rangle_h = \langle \partial_i, \partial_j\rangle_g\,,\ 2 \le i \le j \le n\,, \\ \langle f_*\partial_1, f_*\partial_j\rangle_h = \langle \partial_1, \partial_j\rangle_g = 0\,,\ 2 \le j \le n \\ \langle f_*\partial_1, f_*\partial_1\rangle_h = \langle \partial_1, \partial_1\rangle_g - \langle \partial_1, \partial_1\rangle_{\alpha(x)(dl)^2}\,. \end{cases}$$

Choose a normal vector field $\mathbf{n}$ to $f(\mathbb{R}^n) \subset \mathbb{R}^q$. The map f can be considered as a family of maps $f_p : \mathcal{L}_p \to \mathbb{R}^q$, where $\mathcal{L}_p$ are the leaves of $\mathcal{L}$, and hence we can apply the parametric version of the previous proof using the parametric one-dimensional Lemma 17.5.1. According to property 17.5.1(d), the derivatives $\partial_i\widetilde{f} = \widetilde{f}_*\partial_i\,, i = 2, \dots, n$ of the new map $\widetilde{f}$ can be made arbitrarily close to the respective derivatives $\partial_i f = f_*\partial_i\,, i = 2, \dots, n$, of the initial embedding f. In particular, $\widetilde{f}$ will be an embedding if $\widetilde{f}$ is sufficiently C^0-close to f. On the other hand, the scalar products

$$\langle \widetilde{f}_*\partial_1, \widetilde{f}_*\partial_j\rangle_h\,,\ 2 \le j \le n\,,$$

can be made arbitrarily small by choosing the relations $\widetilde{\mathcal{R}}_{f_p}$ sufficiently close to $\mathcal{R}_{f_p}$. Therefore one can construct $\widetilde{f}$ such that $\widetilde{f}^*h$ will be arbitrarily close to g.

As in the one-dimensional case, for any $\rho > 0$, by choosing $\widetilde{\mathcal{R}}_{f_p}$ sufficiently close to $\mathcal{R}_{f_p}$, we can construct $\widetilde{f}$ such that

$$d_g(f, \widetilde{f}) \le r(g - f^*h, g) + \rho\,.$$

□

21.7. End of the proof of the approximation theorem

Let $g - f^*h = \sum_{i=1,\dots,N} p_i$ be a primitive decomposition of the metric $g - f^*h$. Let

$$g_1 = f^*h + p_1$$
$$g_2 = f^*h + p_1 + p_2$$
$$\dots$$
$$g_N = f^*h + p_1 + \dots + p_N = g\,.$$

Using 21.6.1 we can construct embeddings f_1, f_2, … $f_N = \widetilde{f}$ such that f_i^*h is arbitrarily close to g_i, $i = 1, \dots, N$, and, in particular, $\widetilde{f}^*h = f_N^*h$ is arbitrarily close to $g = g_N$. Moreover, given a constant $\rho > 0$ we can construct embeddings f_1, f_2, … f_N such that for $i = 1, \dots, N$ we have

$$d_{g_i}(f_{i-1}, f_i) < r(p_i, g_i) + \rho'\,,$$

where $\rho' = \rho/N$ and we set $f_0 = f$. Using the inequalities

$$g \geq g_i\,,\ i = 1, \dots, N-1, \ \text{ and } \quad r(p_i, g_i) < 1\,,\ i = 1, \dots, N,$$

and the properties $(r2)$ and $(d1)$ from Section 21.4 we get:

$$\begin{aligned} d_g(f_0, f_1) &\leq d_{g_1}(f_0, f_1) < r(p_1, g_1) + \rho' \leq \\ &\leq r(p_1 + p_2 + \cdots + p_N, g_1 + p_2 + \cdots + p_N) + \rho' = r(\Delta, g) + \rho'\,, \\ d_g(f_1, f_2) &\leq d_{g_2}(f_1, f_2) < r(p_2, g_2) + \rho' \leq \\ &\leq r(p_2 + p_3 + \cdots + p_N, g_2 + p_3 + \cdots + p_N) + \rho' \leq r(\Delta, g) + \rho'\,, \\ &\dots \\ d_g(f_{N-1}, f_N) &\leq d_{g_N}(f_{N-1}, f_N) < r(p_N, g_N) + \rho' \leq r(\Delta, g) + \rho'\,. \end{aligned}$$

On the other hand,

$$d_g(f, \widetilde{f}) = d_g(f_0, f_N) \leq d_g(f_0, f_1) + d_g(f_1, f_2) + \cdots + d_g(f_{N-1}, f_N)\,.$$

Therefore,

$$d_g(f, \widetilde{f}) \ < \ N\, r(\Delta, g) + \rho\,.$$

21.8. Proof of the Nash-Kuiper theorem

Choose constants $\rho_i > 0$ such that $\sum_1^\infty \rho_i < \infty$ and choose a constant $k > 0$ such that $k^2 f^* h > g$.

Fix a decomposition of $\Delta = g - f^*h$ into a sum of N primitive metrics. According to the shortness condition Δ is a family of positive quadratic forms. Fix a positive increasing sequence $\delta_i \to 1$ such that

$$\sqrt{\delta_1} + \sqrt{\delta_2 - \delta_1} + \sqrt{\delta_3 - \delta_2} + \cdots < \infty\,.$$

Note that $g_i = f^*h + \delta_i \Delta \to g$. On the other hand, $f^*h < g_1$ and thus the embedding $f_0 = f : (V, g_1) \to (\mathbb{R}^q, h)$ is strictly short. Using Theorem 21.4.2 we can C^0-approximate the embedding f_0 by an ε_1-isometric embedding $f_1 : (V, g_1) \to (\mathbb{R}^q, h)$ such that

$$\begin{aligned} d(f_0, f_1) &< N r(\delta_1 \Delta, g_1) + \rho_1 = N\sqrt{\delta_1}\, r(\Delta, g_1) + \rho_1 \leq \\ &\leq N\sqrt{\delta_1}\, r(\Delta, f^*g) + \rho_1 = Nk\sqrt{\delta_1}\, r(\Delta, k^2 f^* g) + \rho_1 < \\ &< Nk\sqrt{\delta_1}\, r(\Delta, g) + \rho_1\,. \end{aligned}$$

If ε_1 is sufficiently small then $f_1^*h \approx f_0^*h + \delta_1\Delta$, and hence $f_1^*h < f_0^*h + \delta_2\Delta = g_2$, which means that the embedding $f_1 : (V, g_2) \to (\mathbb{R}^q, h)$ is strictly short. Hence, for any $\varepsilon_2 > 0$ we can C^0-approximate the embedding f_1 by an ε_2-isometric embedding $f_2 : (V, g_2) \to (\mathbb{R}^q, h)$. Moreover, choosing ε_1

sufficiently small at the previous step of the construction we can make the difference $g_2 - f_1^*h$ arbitrarily close to $(\delta_2 - \delta_1)\Delta$. Hence,

$$\begin{aligned} d(f_1, f_2) &< Nr((\delta_2 - \delta_1)\Delta, g_2) + \rho_2 = N\sqrt{\delta_2 - \delta_1}\, r(\Delta, g_2) + \rho_2 \leq \\ &\leq N\sqrt{\delta_2 - \delta_1}\, r(\Delta, f^*g) + \rho_2 = Nk\sqrt{\delta_2 - \delta_1}\, r(\Delta, k^2 f^*g) + \rho_2 < \\ &< Nk\sqrt{\delta_2 - \delta_1}\, r(\Delta, g) + \rho_2\,, \end{aligned}$$

and so on. Note that according to the Remark to Lemma 21.3.1 the constant N does not depend on i.

The sequence $\{f_i\}$ can be chosen C^0-converging to a continuous map $\overline{f}$. On the other hand,

$$\sum_i d(f_i, f_{i+1}) < Nkr(\Delta, g)(\sqrt{\delta}_1 + \sqrt{\delta_2 - \delta_1} + \sqrt{\delta_3 - \delta_2} + \dots) + \sum_i \rho_i < \infty\,.$$

Therefore, according to Lemma 21.4.1 the limit map $\overline{f}$ will automatically be a C^1-smooth isometric embedding and $f_i \xrightarrow{C^1} \overline{f}$.

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